

FLOER THEORY FOR NEGATIVE LINE BUNDLES VIA GROMOV-WITTEN INVARIANTS

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ABSTRACT. We prove that the GW theory of negative line bundles $M = \text{Tot}(L \rightarrow B)$ determines the symplectic cohomology: indeed $SH^*(M)$ is the quotient of $QH^*(M)$ by the kernel of a power of quantum cup product by $c_1(L)$. We prove this also for negative vector bundles and the top Chern class.

We calculate SH^* and QH^* for $\mathcal{O}(-n) \rightarrow \mathbb{CP}^m$. For example: for $\mathcal{O}(-1)$, M is the blow-up of \mathbb{C}^{m+1} at the origin and $SH^*(M)$ has rank m .

We prove Kodaira vanishing: for very negative L , $SH^* = 0$; and Serre vanishing: if we twist a complex vector bundle by a large power of L , $SH^* = 0$.

Observe $SH^*(M) = 0$ iff $c_1(L)$ is nilpotent in $QH^*(M)$. This implies Oancea's result: $\omega_B(\pi_2(B)) = 0 \Rightarrow SH^*(M) = 0$.

We prove the Weinstein conjecture for any contact hypersurface surrounding the zero section of a negative line bundle.

For symplectic manifolds X conical at infinity, we build a homomorphism from $\pi_1(\text{Ham}_\ell(X, \omega))$ to invertibles in $SH^*(X, \omega)$. This is similar to Seidel's representation for closed X , except now they are not invertibles in $QH^*(X, \omega)$.

1. INTRODUCTION

1.1. Gromov-Witten invariants versus Floer cohomology.

The focus of this paper will be symplectic invariants of the total space

$$M = \text{Tot}(\pi_M : L \rightarrow B)$$

of negative (complex) line bundles $L \rightarrow B$ over closed symplectic manifolds (B, ω_B) , although we will show that our techniques work more generally for open symplectic manifolds M conical at infinity which admit Hamiltonian circle actions, and also for negative vector bundles $E \rightarrow B$ (these are not conical at infinity).

By negative line bundle $L \rightarrow B$ we mean that $c_1(L) = -n[\omega_B]$ for real $n > 0$.

Examples: $\mathcal{O}(-n) \rightarrow \mathbb{P}^m$ (classifies negative holomorphic line bundles over \mathbb{P}^m for $n \in \mathbb{Z}$). Duals of ample holomorphic line bundles over compact complex manifolds.

Negativity ensures there is a natural symplectic form ω on M making M conical at infinity (a convexity condition) with $n[\omega_B] \mapsto [\omega]$ via $\pi_M^* : H^2(B) \cong H^2(M)$.

The invariants we will be concerned with are the genus zero Gromov-Witten invariants involved in the construction of the quantum cohomology of M , and the Floer invariants involved in the construction of symplectic cohomology (the natural generalization of Floer homology to open symplectic manifolds M which are conical at infinity, constructed in the exact setup by Viterbo [28] (although there were similar previous incarnations), and in the non-exact setup by the author [20]).

Gromov-Witten invariants are in principle understood for most closed symplectic manifolds, and often they are explicitly calculable thanks to algebraic geometry.

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We suggest Ruan-Tian [22] and McDuff-Salamon [15] as references. We will be concerned with genus zero GW invariants of the (non-closed) M and of certain Hamiltonian fibrations over \mathbb{P}^1 with fibre M . The first arise in algebraic geometry as the *twisted Gromov-Witten invariants* of (B, L) and were studied by Coates-Givental [5] and Lee [14]: essentially the GW theory of M is determined by the GW theory of B and the invariant $c_1(L)$. The second are known for closed symplectic manifolds by the work of Seidel [26], and we succeeded in generalizing these to the open setup despite the difficulties caused by the non-compactness.

Floer invariants, on the other hand, are notoriously difficult to calculate explicitly because the chain differential comes from counting solutions of certain elliptic partial differential equations which require a *generic* choice of ω -compatible almost complex structure J on M . In practice, this means that one always has to perturb a given J , so one cannot compute anything unless things vanish for grading reasons.

For symplectic cohomology, the difficulty of computing the invariants is even more dramatic, because they arise as a direct limit of Floer cohomologies:

$$SH^*(X, \omega_X) = \varinjlim HF^*(H, \omega_X)$$

involving Hamiltonians $H : X \rightarrow \mathbb{R}$ which are “linear” at infinity, and the connecting maps $HF^*(H_1, \omega_X) \rightarrow HF^*(H_2, \omega_X)$ are Floer continuation maps which increase the slope at infinity. These continuation maps, again solutions of an elliptic PDE, can almost never be computed explicitly for the same reasoning.

This phenomenon is apparent in the literature, where known computations involve showing vanishing results by indirect grading/action tricks (for example, for $X = \mathbb{C}^m$ and generally $X =$ subcritical Stein manifold [7]). For this reason, a precious guide to proving non-vanishing of $SH^*(X)$ *a posteriori* has been by detecting submanifolds which obstruct vanishing (for example, when $\omega_X = d\theta$ is exact, and X contains an exact Lagrangian submanifold [28]). Other attempts involve proving that $SH^*(X)$ reduces to a topological invariant by continuation arguments, again not explicitly computable (for instance, various versions of Viterbo’s result [28] that $SH^*(T^*B, d\theta) \cong H_{\dim_{\mathbb{C}}(B)-*}(\mathcal{LB})$, where \mathcal{LB} is the free loop space, but where we do not know what the isomorphism actually is).

It comes therefore as a surprise that for $M = \text{Tot}(L \rightarrow B)$ we will calculate the Floer cohomologies and the continuation maps explicitly and directly, by transforming the Floer theoretic problem into an essentially algebraic-geometric problem in Gromov-Witten theory. It is also surprising that we will explicitly recover the ring structure on symplectic cohomology. Finally we emphasize that the setup we are in is very novel for symplectic cohomology literature: we are in a highly non-exact setup (the zero section is a symplectic submanifold and holomorphic spheres play a crucial role, unlike the much studied setup of exact cotangent bundles) and for $\dim_{\mathbb{C}} B > 1$ we are dealing with manifolds which do not admit a Stein structure.

Theorem 1. *Let M be the total space of a negative line bundle $L \rightarrow (B, \omega_B)$ (satisfying a weak⁺ monotonicity condition). Then for $k \geq \dim H^*(B)$,*

$$SH^*(M) \cong QH^*(M) / \ker r^k$$

where $r : QH^*(M) \rightarrow QH^{*+2}(M)$ is the (non-invertible) endomorphism given by quantum cup product by the first Chern class $r(1) = \pi_M^* c_1(L) \in QH^2(M)$.

The isomorphism is induced by $c^* : QH^*(M) \rightarrow SH^*(M)$: a canonical algebra homomorphism identifiable with r^k . The induced action of r on $SH^*(M)$ is

$$\boxed{\mathcal{R} : SH^*(M) \rightarrow SH^{*+2}(M)}$$

a degree 2 automorphism which on the chain level is a natural rotation action $\mathcal{S} : HF^*(H, J) \rightarrow HF^{*+2}(g^*H, g^*J)$ determined by the loop $g = (e^{2\pi it})_{t \in S^1}$ of Hamiltonian diffeomorphisms which rotate the fibres of L .

Corollary. *c^* is never an isomorphism. So any contact hypersurface in M surrounding the zero section contains a closed Reeb orbit (Weinstein conjecture).*

Proof. $1 \notin \text{im } r$ as it would involve a GW invariant with an evaluation condition with the point class, which can be moved to infinity. So $\ker r \neq 0$, so $\ker c^* \neq 0$. The rest is a standard consequence of the construction and invariance of SH^* . \square

Corollary 2. *$SH^*(M) = 0 \Leftrightarrow \pi_M^* c_1(L)$ is nilpotent in $QH^*(M)$. So if the quantum cup product reduces to the ordinary cup product, then $SH^*(M) = 0$.*

Theorem. *For negative vector bundles $E \rightarrow B$, the analogue of the Theorem holds: r is a degree $2\text{rank}_{\mathbb{C}} E$ endomorphism given by quantum cup product by the top Chern class $r(1) = \pi_M^* c_{\text{rank}_{\mathbb{C}}}(E)$, and \mathcal{R} is a degree $2\text{rank}_{\mathbb{C}} E$ automorphism on $SH^*(\text{Tot}(E))$. In particular if $\text{rank}_{\mathbb{C}} E > \dim_{\mathbb{C}} B$ then $SH^*(\text{Tot}(E)) = 0$.*

1.2. How r arises algebro-geometrically and Floer theoretically.

Algebro-geometrically the map r arises from 2-pointed genus 0 Gromov-Witten invariants counting sections of the Hamiltonian fibration $E_g \rightarrow \mathbb{P}^1$ with fibre M , constructed from the loop of rotations $g_t = e^{2\pi it}$ by the clutching construction. Heuristically r is the pull-push map

$$H^*(M) \rightarrow H^{*+2}(M), \quad a \mapsto \sum_{\beta \in H_2(M)} (\text{ev}_{z_\infty})_! (\text{ev}_{z_0}^*(a) \wedge e(\text{Obs}_\beta))$$

where Obs_β is the obstruction bundle over the moduli space

$$\mathcal{M}_\beta = \overline{\mathcal{M}}_{0,2}(E_g, [\mathbb{P}^1] + (j_{z_0})_*\beta)$$

of stable maps u from 2-pointed genus 0 nodal curves to E_g representing the class $[\mathbb{P}^1] + (j_{z_0})_*\beta$ where $[\mathbb{P}^1] \in H_2(E_g)$ is the base of E_g and j_{z_0} is inclusion of the fibre at the South Pole $z_0 \in \mathbb{P}^1$. Composing with the projection $\pi_g : E_g \rightarrow \mathbb{P}^1$, the main component of u yields an isomorphism to \mathbb{P}^1 . So u can be viewed as a holomorphic section of E_g possibly with holomorphic bubbles in the fibres (killing the $PSL(2, \mathbb{C})$ reparametrization freedom by making it a section). The two maps $\text{ev} : \mathcal{M}_\beta \rightarrow M$ are evaluation of sections of E_g at the two Poles $z_0, z_\infty \in \mathbb{P}^1$.

More precisely, r is a Novikov-weighted count of the zero dimensional moduli spaces of pseudo-holomorphic sections of $E_g \rightarrow \mathbb{P}^1$ which intersect a given locally finite quantum cycle in the fibre over z_0 and a given quantum cycle over z_∞ .

Floer theoretically, the map r is the composite

$$\begin{array}{ccccc} HF^*(H_0, J, \omega) & \xrightarrow{\mathcal{S}} & HF^{*+2}(g^*H_0, g^*J, \omega) & \xrightarrow{\varphi_0} & HF^{*+2}(H_0, J, \omega) \\ \uparrow \psi^- & & \searrow \mathcal{R} & & \downarrow \psi^+ \\ QH^*(M, \omega) & \xrightarrow{\quad r \quad} & QH^{*+2}(M, \omega) & & \end{array}$$

where $H_0 : M \rightarrow \mathbb{R}$ is a Hamiltonian of “slope zero” at infinity (more precisely: whose positive slope decays to 0 at infinity), so it gives rise to identifications ψ^\pm between the Floer complexes and the (quantum) Morse chain complexes; where \mathcal{S} is the natural isomorphism at the chain level induced by identifying the relevant Floer moduli spaces by pulling back the data H_0, J via g ; and where φ_0 is a Floer continuation map obtained by homotopying the data.

Theorem. *The algebro-geometrical and the Floer theoretical construction of r agree, that is the above diagram commutes.*

This result, and Theorem 1, can be heuristically viewed as a symplectic analogue of the quantum Lefschetz hyperplane theorem [14]: the invariants of the hyperplane section $B \subset M$ are recovered from invariants of the ambient M and a quantum multiplication operation by an Euler class.

The difficulty in relating the two constructions of r (compared with a similar setup in the closed case due to Seidel [26]) involves the fact that we are using *non-compact* Hamiltonian fibrations and *non-monotone* homotopies (arising in ψ^+).

1.3. The role of r in determining $SH^*(M)$.

The symplectic cohomology of M arises as a direct limit of Floer cohomologies

$$QH^*(M) \begin{matrix} \xrightarrow{\psi^-} \\ \xleftarrow{\psi^+} \end{matrix} HF^*(H_0) \xrightarrow{\varphi_1} HF^*(H_1) \xrightarrow{\varphi_2} HF^*(H_2) \xrightarrow{\varphi_3} \dots$$

where H_i are carefully chosen Hamiltonians with slope proportional to i , the φ_i are continuation maps. The direct limit of the composition of those maps defines $c^* : QH^*(M) \rightarrow SH^*(M)$. We prove in 2.13 that the ψ^\pm are identifications of algebras and that c^* is a Λ -algebra homomorphism (using a Novikov ring Λ).

After suitable identifications, we prove in 4.2 that the above sequence becomes:

$$V = V \xrightarrow{\varphi} V \xrightarrow{\varphi} V \xrightarrow{\varphi} \dots$$

where $V = QH^*(M)$ and φ is quantum cup product by $r(1)$. This involves a special choice of H_i : recall $H = m\pi|z|^2$ on \mathbb{C} for $m \notin \mathbb{Z}$ only has Hamiltonian 1-orbit 0, and in our case the cohomology of the zero section plays the role of this 1-orbit 0.

By linear algebra, $\varphi^k(V)$ stabilizes for $k \geq \text{rank}_\Lambda QH^*(M)$ and $\varphi^k(V) \cong V / \ker \varphi^k$. Say it stabilizes at stage k . Then φ is an automorphism on $\varphi^k(V)$. In the direct limit, we identify $v \sim \varphi(v)$, so $SH^*(M)$ can be identified as a Λ -vector space to $\varphi^k(V) \subset HF^*(H_k)$, and $\varphi^k : V \rightarrow \varphi^k(V)$ can be identified with c^* .

Thus c^* is surjective and $\ker c^* = \ker r^k$. Since c^* is an algebra homomorphism, it induces the quotient isomorphism of Λ -algebras

$$SH^*(M) \cong QH^*(M) / \ker c^* = QH^*(M) / \ker r^k,$$

proving Theorem 1. The product structure is discussed in more detail in 4.3.

The reason the sequence simplifies so dramatically, is that conjugation by the rotation $\mathcal{S} : HF^*(H_i) \rightarrow HF^{*+2}(H_{i-1})$ recovers all φ_i from φ_0 : $\varphi_i = \mathcal{S}^{-i} \varphi_0 \mathcal{S}^i$. So $SH^*(M)$ is determined via linear algebra by a map

$$QH^*(M, M \setminus B) \rightarrow QH^*(M)$$

corresponding to $HF^*(-H_0) \rightarrow HF^*(H_0)$ (identifiable with φ_0). Up to first approximation, this map is the natural map for the pair $(M, M \setminus B)$, which in the Gysin sequence for the sphere bundle of L corresponds to ordinary cup product

$H^*(B) \rightarrow H^{*+2}(B)$ by $c_1(L)$. The surprising result is that there are quantum correction terms in the Floer continuation map, and this first approximation equals the continuation map of Morse cohomologies (the Floer complexes for small $\pm H_0$ reduce to Morse complexes). This is unlike the exact setup [21] or the setup $\omega_B(\pi_2(B)) = 0$, in which by arguments à la Salamon-Zehnder [24] for a homotopy of C^2 -small time-independent Morse Hamiltonians the Floer continuation map reduces to the Morse continuation map (solutions become time-independent).

1.4. Non-vanishing of symplectic cohomology of the blowup of \mathbb{C}^{m+1} .

Corollary 3. *For $\mathcal{O}(-1) \rightarrow \mathbb{P}^m$, $SH^*(M)$ has rank m . Indeed as Λ -algebras:*

$$\begin{aligned} QH^*(M) &= \Lambda[\omega_Q]/(\omega_Q^{m+1} + t \cdot \omega_Q) \\ SH^*(M) &\cong \Lambda[\omega_Q]/(\omega_Q^m + t \cdot 1) \end{aligned}$$

($\Lambda = \text{Novikov field}$, $\omega_Q = \pi_M^* \omega_{\mathbb{P}^1} \otimes 1 \in QH^2(M)$, ω_Q^m are quantum powers).

Recall that the M of the Corollary arise as the blow-up of \mathbb{C}^{m+1} at the origin. So the symplectic cohomology has changed under blow-up as $SH^*(\mathbb{C}^{m+1}) = 0$. Interestingly the growth-rate [25, Sec.(4a)] of $SH^*(M)$ is 0 despite $SH^*(M) \not\cong QH^*(M)$. When this non-isomorphism occurs, there is a non-constant Hamiltonian orbit, and one typically expects its iterates to force $\dim_\Lambda SH^*(M) = \infty$.

Remark 4 (Smith). *Ivan Smith discovered an essential torus in $\text{Tot}(\mathcal{O}(-1) \rightarrow \mathbb{P}^1)$ in [27, Corollary 4.22]: the sphere bundle lying over the equator of \mathbb{P}^1 of constant radius making the torus monotone. Essential tori are defined in [25, Sec. (5b)]. In [25, Prop.5.2], Seidel-Smith state that if a 4-dimensional (exact) Liouville domain $(M, d\theta)$ contains an essential Lagrangian torus then $SH^*(M, d\theta) \neq 0$. The proof is not written down in the literature in detail, but it is briefly sketched in Seidel [25, Sec.(5b)]. If one assumes that this result holds also for monotone essential tori in non-exact 4-dimensional symplectic M conical at infinity, then the presence of the essential torus would imply a posteriori that $SH^*(\text{Tot}(\mathcal{O}(-1) \rightarrow \mathbb{P}^1)) \neq 0$.*

1.5. Non-vanishing of symplectic cohomology of $M = \text{Tot}(\mathcal{O}(-n) \rightarrow \mathbb{P}^m)$.

By using virtual localization techniques similar to Kontsevich [13] and Graber-Pandharipande [10], but adapted to the setup of holomorphic sections of E_g , we determine r explicitly for $n < 1 + \frac{m}{2}$, and determine enough about r for $n < 1 + m$:

Theorem 5. *Let $M = \text{Tot}(\mathcal{O}(-n) \rightarrow \mathbb{P}^m)$, $N = 1 + m - n$. In characteristic 0:*

$1 \leq n < 1 + \frac{m}{2}$	$\begin{aligned} QH^*(M) &= \Lambda[\omega_Q]/(\omega_Q^{1+m} + n^n t \omega_Q^n) \\ SH^*(M) &= \Lambda[\omega_Q]/(\omega_Q^N + n^n t) \end{aligned}$
$1 + \frac{m}{2} \leq n < 1 + m$	$SH^*(M) \neq 0$ has rank a multiple of N
$n = 1 + m$	$SH^*(M) = 0$ (borderline case: $c_1(TM) = 0$)
$2 + m \leq n \leq 2m$	M does not satisfy weak ⁺ monotonicity
$n > 2m$	$\begin{aligned} QH^*(M) &= \Lambda[\omega_Q]/(\omega_Q^{1+m}) \text{ is ordinary} \\ SH^*(M) &= 0 \end{aligned}$

Over characteristic 2, this also holds, except $SH^*(M) = 0$ for even n .

1.6. The aspherical case: $\omega_B(\pi_2(B)) = 0$.

Negative line bundles satisfying $\omega_B(\pi_2(B)) = 0$ have been studied by Oancea in his Ph.D. thesis (see [16]). This involves a difficult construction of a Leray-Serre spectral sequence for symplectic cohomology, which immediately collapses for negative line bundles since fibres have $SH^*(\mathbb{C}) = 0$, and so $SH^*(M) = 0$. Observe that Corollary 2 gives a new proof of this result: when $\omega_B(\pi_2(B)) = 0$ then $\omega(\pi_2(M)) = 0$ so quantum cup product on M is ordinary cup product.

Because of this vanishing result, Oancea conjectured that vanishing should hold for any negative line bundle even without the condition $\omega_B(\pi_2(B)) = 0$.

Corollary 3 shows this conjecture is not true. It also shows there cannot be a spectral sequence $E_2^{p,q} = QH^p(B, \mathcal{SH}^q(\mathbb{C}))$ converging to $SH^*(M)$. So $\omega(\pi_2(M)) = 0$ is more than a technical assumption, which is surprising in Floer theory.

We also have to point out that the assumption $\omega_B(\pi_2(B)) = 0$ is extremely restrictive: it excludes all simply connected B , and it excludes any complex variety B which contains a holomorphic \mathbb{P}^1 . However it holds for surfaces of genus ≥ 1 .

1.7. The Calabi-Yau type case: $c_1(TM)(\pi_2(M)) = 0$.

Theorem 6. *If $c_1(TM)(\pi_2(M)) = 0$, then $SH^*(M) = 0$.*

Proof 1. Λ has grading 0, so $(\pi_M^* c_1(L))_Q^{\dim_{\mathbb{C}} B+1} \in H^{\dim_{\mathbb{C}} B+1}(M) \otimes \Lambda = 0$. \square

Proof 2. $SH^*(M)$ is \mathbb{Z} -graded. Rotation in the fibres induces $\mathcal{S} : HF^*(H_i) \xrightarrow{\sim} HF^{*+2}(H_{i-1})$. In the direct limit, $\mathcal{S} : SH^*(M) \rightarrow SH^{*+2}(M)$ is an automorphism. So $SH^*(M)$ is 2-periodic, so it is either 0 or ∞ -dimensional. But $\text{rank}_{\Lambda} HF^*(H_i) = \text{rank}_{\Lambda} QH^*(M)$, so $\text{rank}_{\Lambda} SH^*(M) \leq \text{rank}_{\Lambda} QH^*(M)$, so $SH^*(M) = 0$. \square

For example, this applies to $\mathcal{O}(-(1+m)) \rightarrow \mathbb{P}^m$. More generally, let B be a *Fano variety*: a closed complex manifold with ample anticanonical bundle \mathcal{K}^\vee , where $\mathcal{K} = \Lambda_{\mathbb{C}}^{\text{top}} T^*B$. Since $c_1(TB) = -c_1(\mathcal{K})$, and in general $c_1(TM) \equiv c_1(TB) + c_1(L)$ (via the identification $H^2(M) \cong H^2(B)$), we deduce:

Example. *Let $L = \text{canonical bundle } \mathcal{K} \rightarrow \text{Fano variety } B$. Then $SH^*(M) = 0$.*

Example. *Hyperkähler ALE spaces (minimal resolutions of simple singularities \mathbb{C}^2/Γ) are not total spaces of negative line bundles (except for T^*S^2 which is $\mathcal{O}(-2) \rightarrow \mathbb{P}^1$), but they admit a circle action g similar to rotation in the fibres for $\omega = \omega_I$ (see [20]). Since $c_1(\text{ALE space}) = 0$, we deduce $SH^*(\text{ALE space}, \omega_I) = 0$.*

1.8. The role of weak⁺ monotonicity.

Because of 1.6 and 1.7, one is really interested in the case $\omega_B(\pi_2(B)) \neq 0$ and $c_1(TM)(\pi_2(M)) \neq 0$. This causes two difficulties in Floer homology: (1) the action functional which defines the chain differential becomes multivalued and bubbling phenomena can occur; (2) Floer homology is only $\mathbb{Z}/2N$ -graded where

$$N\mathbb{Z} = c_1(TM)(\pi_2(M)) = (c_1(TB) + c_1(L))(\pi_2(B)).$$

A standard machinery due to Hofer-Salamon [12] ensures Floer homology can be defined if we assume that M is **weak**, meaning **at least one of**:

- (1) $\omega(\pi_2(M)) = 0$ or $c_1(TM)(\pi_2(M)) = 0$,
- (2) M is monotone: $\exists \lambda > 0$ such that $\omega(A) = \lambda c_1(TM)(A)$ for all $A \in \pi_2(M)$,
- (3) the minimal Chern number $|N| \geq \dim_{\mathbb{C}} B$.

1.9. The rank of $SH^*(M)$.

Corollary 7. *For weak M ,*

- (1) $\text{rank}_\Lambda SH^*(M) < \text{rank } H^*(B)$;
- (2) $\text{rank}_\Lambda SH^*(M)$ is a multiple of $|N|$.
- (3) if $|N| \geq \text{rank } H^*(B)$ then $SH^*(M) = 0$.

Proof. (1): follows by Theorem 1. (2): $SH^*(M)$ is $\mathbb{Z}/2N$ -graded, Λ is generated by elements in degrees $\in 2N\mathbb{Z}$ so they preserve the grading of SH^* . So the automorphism \mathcal{R} induces $SH^0(M) \cong SH^2(M) \cong \dots \cong SH^{2|N|-2}(M)$. Similarly for odd pieces. So $\text{rank}_\Lambda SH^*(M) = |N| \cdot (d_0 + d_1)$, $d_i = \text{rank}_\Lambda SH^i(M)$. (3) follows. \square

Example. $SH^*(M) = 0$ for $\mathcal{O}(-n) \rightarrow \mathbb{P}^m$ if $n \geq 2m + 2$ since $|N| \geq 1 + m$.

1.10. Kodaira Vanishing for $SH^*(M)$.

Theorem 8. *If the line bundle $L \rightarrow B$ is sufficiently negative then quantum cup product on M is ordinary cup product, so $SH^*(M) = 0$ by Corollary 2.*

Proof 1. $\omega * \omega^j = \sum \text{GW}_{0,3,\beta}^M(\text{PD}(\omega), \text{PD}(\omega^j), 2\ell\text{-cycle}) [2\ell\text{-form}] \otimes \beta$, summing over $\ell = 1 + j - c_1(TM)(\beta)$ and over appropriate forms/cycles. Now

$$c_1(TM)(\beta) = c_1(TB)(\pi_M \circ u) + c_1(L)(\pi_M \circ u),$$

and since $\pi_M \circ u$ is holomorphic: $c_1(L)(\pi_M \circ u) = -n\omega_B(\pi_M \circ u) < 0$ unless u is constant (if $\pi_M \circ u$ is constant then by the maximum principle u is constant). So for non-constant u : $n \gg 0$ implies $c_1(TM)(\beta) \ll 0$ so $\ell \gg 0$ so there are no 2ℓ -forms. So only constants contribute, which yield ordinary cup product. \square

Proof 2. Let $N_{\text{eff}}\mathbb{Z} = c_1(TM)\{\text{pseudo-holomorphic } v : \mathbb{P}^1 \rightarrow B \subset M\}$, and $\Lambda_{\text{eff}} \subset \Lambda$ the subring generated by “effective” $\pi_2(M)$ -classes i.e. arising as such $[v]$. Now $r(1)$ and quantum product involve $(\text{forms} \otimes [v])$, so restrict $r \in \text{End}_{\Lambda_{\text{eff}}}(H^*(M) \otimes \Lambda_{\text{eff}})$. N_{eff} grows proportionally to n (since $\omega_B(v) > 0$ unless v is constant). So for $n \gg 0$, $N_{\text{eff}} \geq \max(\text{rank } H^*(M), \dim_{\mathbb{C}} B)$, so the characteristic polynomial of r yields a linear dependence among $r(1), r(1)^2, \dots, r(1)^{|N_{\text{eff}}|}$, but these lie in different degrees (Λ_{eff} is in degrees $2N_{\text{eff}}\mathbb{Z}$), so some $r(1)^k = 0$, so $SH^*(M) = 0$ by Corollary 2. \square

Examples. For a K3 surface B , $\omega_B \in H^2(B; \mathbb{Z})$, and $n \geq 24$ then $SH^*(M) = 0$. For $L = \mathcal{K}^{k+1} \rightarrow \text{Fano } B$ with $k \geq \max(\text{rank } H^*(B), \dim_{\mathbb{C}} B)$, $SH^*(M) = 0$.

Corollary 9. *If $E \rightarrow B$ is any line bundle, and $L \rightarrow B$ is any negative line bundle, then $M_k = \text{Tot}(E \otimes L^{\otimes k} \rightarrow B)$ is weak for $k \gg 0$ and $SH^*(M_k) = 0$ for $k \gg 0$.*

Proof. Hermitian connections on L, E induce one on $E \otimes L^{\otimes k}$ with curvature $\mathcal{F}^E + k\mathcal{F}^L$. Then $\frac{1}{2\pi i}k\mathcal{F}^L(v, J_B v) + \frac{1}{2\pi i}\mathcal{F}^E(v, J_B v) < 0$ for $k \gg 0$ by making the first term dominate (see Lemma 38: we pick a suitable connection on L). Hence $E \otimes L^{\otimes k}$ is a negative line bundle with $n \gg 0$ if $k \gg 0$ (see the comment after Lemma 38). \square

Remark. *Strictly speaking, weakness may not be satisfied by M_k in case (3) of 1.8 if $|N| < \dim_{\mathbb{C}} B$. But since $|N_{\text{eff}}| \geq \dim_{\mathbb{C}} B$, all Floer theoretic issues such as bubbling can be avoided: only effective $\pi_2(M)$ -classes are involved in these issues.*

1.11. Negative vector bundles and Serre Vanishing. Complex vector bundles $E \rightarrow B$ are negative if a suitable negative curvature condition holds (Definition 66). The automorphism $\mathcal{R} : SH^*(M) \rightarrow SH^{*+2\text{rank}_{\mathbb{C}} E}(M)$ implies $2\text{rank}_{\mathbb{C}} E$ -periodicity in ranks so Corollary 7 becomes: for weak $M = \text{Tot}(E \rightarrow B)$,

- (1) $\text{rank}_{\Lambda} SH^*(M) < \text{rank } H^*(B)$;
- (2) $\text{rank}_{\Lambda} SH^*(M)$ is a multiple of $|N|/\text{rank}_{\mathbb{C}} E$, the multiple is $d_0 + d_1 + \dots + d_{2\text{rank}_{\mathbb{C}} E-1}$ where $d_i = \text{rank}_{\Lambda} SH^i(M)$.
- (3) if $|N| \geq \text{rank}_{\mathbb{C}} E \cdot \text{rank } H^*(B)$ then $SH^*(M) = 0$.

Theorem 6 and Corollary 2 (for $\pi_M^* c_{\text{rank}_{\mathbb{C}} E}(E)$) hold for the same reasons.

Theorem 10. *Let $E \rightarrow B$ be any complex vector bundle, and $L \rightarrow B$ a negative line bundle. Then for $k \gg 0$: $M_k = \text{Tot}(E \otimes L^{\otimes k} \rightarrow B)$ is a negative vector bundle and is weak. Also $SH^*(M_k) = 0$ for $k \gg 0$.*

Proof. As in 1.10, using $c_1(TM_k) \equiv c_1(TB) + c_1(E) + kc_1(L)$. In Proof 2, now need $|N_{\text{eff}}| \geq \text{rank}_{\mathbb{C}} E \cdot \text{rank } H^*(B)$ and (for weakness) $|N_{\text{eff}}| \geq \dim_{\mathbb{C}} B + \text{rank}_{\mathbb{C}} E - 1$. \square

1.12. The general theory: a representation of $\pi_1(\text{Ham}_{\ell}(X, \omega))$ on $SH^*(X)$.

Let (X, ω) be any symplectic manifold conical at infinity satisfying weak⁺ monotonicity (2.1). So X has the form $\Sigma \times [1, \infty)$ at infinity, with coordinate $R \in [1, \infty)$.

Denote $\text{Ham}_{\ell}(X, \omega)$ the Hamiltonian diffeomorphisms generated by Hamiltonians K which at infinity have “linear growth”:

$$m_1 R + c_1 \leq K \leq m_2 R + c_2,$$

some $m_i, c_i \in \mathbb{R}$ (for example if K only depends on R and is linear). Write $\text{Ham}_{\ell \geq 0}(X, \omega), \text{Ham}_{\ell > 0}(X, \omega)$ for the subsets involving $m_1 \geq 0, m_1 > 0$.

For $g : S^1 \rightarrow \text{Ham}_{\ell}(X, \omega)$, there is a group Γ of choices of “lifts” \tilde{g} related to the Novikov ring Λ (3.1). These define an extension $\widetilde{\pi}_1(\text{Ham}_{\ell}(X, \omega))$ of $\pi_1(\text{Ham}_{\ell}(X, \omega))$.

Lemma. $g : S^1 \rightarrow \text{Ham}_{\ell}(X, \omega)$ gives rise to automorphisms $\mathcal{S}_{\tilde{g}} \in \text{Aut}(SH^*(X))$ given by pair-of-pants product by $\mathcal{S}_{\tilde{g}}(1) \in SH^{2I(\tilde{g})}(X)$. There is a homomorphism:

$$\boxed{\widetilde{\pi}_1(\text{Ham}_{\ell}(X, \omega)) \rightarrow SH^*(X)^{\times}, \quad \tilde{g} \mapsto \mathcal{S}_{\tilde{g}^{-1}}(1)}$$

Theorem. Any $g : S^1 \rightarrow \text{Ham}_{\ell > 0}(X, \omega)$ gives rise to Λ -algebra automorphisms $\mathcal{R}_{\tilde{g}} = \mathcal{S}_{\tilde{g}} : SH^*(X) \rightarrow SH^{*+2I(\tilde{g})}(X)$ making the following diagram commute:

$$\boxed{\begin{array}{ccc} SH^*(X) & \xrightarrow[\sim]{\mathcal{R}_{\tilde{g}}} & SH^{*+2I(\tilde{g})}(X) \\ \uparrow c^* & & \uparrow c^* \\ QH^*(X) & \xrightarrow{r_{\tilde{g}}} & QH^{*+2I(\tilde{g})}(X) \end{array}}$$

$r_{\tilde{g}}$ is a count of holomorphic sections of a Hamiltonian fibration $E_{\tilde{g}} \rightarrow S^1$, it is quantum cup product by $r_{\tilde{g}}(1) \in QH^{2I(\tilde{g})}(X)$, and via ψ^{\pm} it can be identified with $\mathcal{R}_{\tilde{g}} = \varphi \circ \mathcal{S}_{\tilde{g}} : HF^*(H_0) \rightarrow HF^{*+2I(\tilde{g})}(H_0)$ where φ is a continuation.

Proof. The maps $\mathcal{R}_{\tilde{g}} = \varphi_H \circ \mathcal{S}_{\tilde{g}} : HF^*(H) \rightarrow HF^*(g^*H) \rightarrow HF^*(H)$, where φ_H is the continuation, are compatible with continuations since $\mathcal{S}_{\tilde{g}}$ is (Theorem 18) and continuations are. The identification of $r_{\tilde{g}}$ is a gluing argument (Theorem 35). \square

Corollary. For any $g : S^1 \rightarrow \text{Ham}_{\ell > 0}(X, \omega)$,

$$\boxed{SH^*(X) \cong QH^*(X) / \ker r_g^k}$$

is induced by $c^* : QH^*(X) \rightarrow SH^*(X)$ for $k \geq \text{rank } H^*(X)$.

Proof. For H_0 small, $HF^*(H_0) \cong QH^*(X)$ (it reduces to the Morse complex). There is a natural pull-back $H_k = (g^k)^* H_0$ (see 3.2). The Hamiltonian generating g has positive linear growth, so $SH^*(X) = \varinjlim HF^*(H_k)$ (Remark 19). $S_g^k : HF^*(H_k) \cong HF^{*+2kI(\tilde{g})}(H_0)$, so as in 1.3: $SH^*(X) \cong HF^*(H_0) / \ker \mathcal{R}_{\tilde{g}}^k$. \square

The element $r_{\tilde{g}}(1)$ plays a similar role to the quantum invertible element $q(g, \tilde{g})$ of the Seidel representation [26] for closed symplectic manifolds (C, ω) :

$$q : \widetilde{\pi_1}(\text{Ham}(C, \omega)) \rightarrow QH_*(C, \omega)^\times, \quad g \mapsto q(g, \tilde{g}).$$

These invertibles arise naturally in Floer homology and one can pass to quantum homology via $QH_*(C) \cong HF_*(C)$. In our case there is only a homomorphism $c^* : QH^*(X) \rightarrow SH^*(X)$, the $r_{\tilde{g}}(1)$ can be non-invertible in $QH^*(X)$, but they become invertibles $\mathcal{R}_{\tilde{g}}(1)$ on the quotient $SH^*(X)$. Indeed, $r_{\tilde{g}}$ represents the continuation maps defining $SH^*(X)$ and $r_{\tilde{g}}$ is nilpotent precisely when $SH^*(X) = 0$.

The natural generalization of the Seidel representation to non-compact (X, ω) would have been to consider compactly supported Hamiltonian diffeomorphisms $\text{Ham}_{\ell=0}(X, \omega)$, so that their action does not affect the dynamics at infinity. In that case, $r_{\tilde{g}}(1) = \text{PD}[q(g, \tilde{g})]$ is an invertible in $QH^*(X, \omega)$ in degree 0 (Example 16), and it induces a degree preserving automorphism $\mathcal{R}_{\tilde{g}}$ on $SH^*(X, \omega)$.

However, it would not have helped to compute $SH^*(X, \omega)$. To help compute $SH^*(X)$ we need the diffeomorphism to dramatically affect the dynamics at infinity, so that it relates the different Floer cohomologies arising in the direct limit.

1.13. Outline of the paper, Conventions, Acknowledgements.

Outline of the paper.

Section 2: review of HF^*, SH^*	Section 7: negative line bundles
Section 3: $\mathcal{S}_{\tilde{g}}$ and $\pi_1 \text{Ham}(M, \omega)$ action on Floer complexes	Section 8: $SH^*(\mathcal{O}(-n) \rightarrow \mathbb{P}^1)$ by a Grothendieck-Riemann-Roch argument
Section 4: $\mathcal{R}_{\tilde{g}}$ and the Floer theoretic $r_{\tilde{g}}$	Section 9: $QH^*(\mathcal{O}(-1) \rightarrow \mathbb{P}^m)$ directly and $SH^*(\mathcal{O}(-n) \rightarrow \mathbb{P}^m)$ by virtual localization
Section 5: $E_g \rightarrow \mathbb{P}^1$ and the algebro-geometric $r_{\tilde{g}}$	Section 10: $r(1) = \pi_M^* c_1(L)$ for negative $L \rightarrow B$
Section 6: review GW invariants	Section 11: negative vector bundles.

Conventions. We only consider the summand of $SH^*(M)$ coming from the contractible orbits (which is everything if $\pi_1(B) = 1$). Observe that if $\pi_1(B) \neq 1$, then a vanishing result for this summand implies vanishing of the full $SH^*(M)$ since the unit lies in this summand (Corollary 14). We use $\text{char}(\Lambda) = 2$ to avoid discussing orientations, but we kept track of orientation signs: Remark 65.

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2. CONICAL SYMPLECTIC MANIFOLDS AND SYMPLECTIC COHOMOLOGY

2.1. Conical symplectic manifolds. We will consider non-compact symplectic manifolds (M, ω) , whose symplectic form ω is typically non-exact. We call M **conical at infinity** if outside a bounded domain $M_0 \subset M$ there is a symplectomorphism

$$\psi : (M \setminus M_0, \omega|_{M \setminus M_0}) \cong (\Sigma \times [1, \infty), d(R\alpha)).$$

where (Σ, α) is a contact manifold, and R is the coordinate on $[1, \infty)$.

The conical condition implies that outside of M_0 the symplectic form becomes exact: $\omega = d\theta$ where $\theta = \psi^*(R\alpha)$. It also implies that the Liouville vector field $Z = \psi^*(R\partial_R)$ (defined by $\omega(Z, \cdot) = \theta$) will point strictly outwards along ∂M_0 . Finally, it implies that ψ is induced by the flow of Z for time $\log R$, so we can simply write $\Sigma = \partial M_0$, $\alpha = \theta|_\Sigma$ (pull-back).

By *conical structure* $J = J_t$ we mean a (typically time-dependent) ω -compatible almost complex structure on M (so $\omega(\cdot, J\cdot)$ is a J -invariant metric) satisfying the *contact type condition* $J^*\theta = dR$ for large R . On Σ this implies $JZ = Y$ where Y is the Reeb vector field for (Σ, α) defined by $\alpha(Y) = 1$, $d\alpha(Y, \cdot) = 0$.

By choosing α or Σ generically, one ensures that α is sufficiently generic so that the periods of the Reeb vector field Y form a countable closed subset of $[0, \infty)$.

In this Section we succinctly construct $SH^*(M)$. In the exact setup ($\omega = d\theta$ on all of M) symplectic cohomology was introduced by Viterbo [28], and we refer to [19] for details and to Seidel [25] for a survey. In the non-exact setup it was first constructed by the author in [20], to which we refer for details. In this paper we use a larger Novikov ring than in [20] (see 2.6), so that our conventions mirror [12, 25].

2.2. Weak⁺ monotonicity. We assume M satisfies **at least one of**:

- (1) there is a $\lambda \geq 0$ such that $\omega(A) = \lambda c_1(TM)(A)$ for all $A \in \pi_2(M)$;
- (2) $c_1(TM)(A) = 0$ for all $A \in \pi_2(M)$;
- (3) the minimal Chern number $|N| \geq \dim_{\mathbb{C}} M - 1$.

Recall $|N|$ is defined by $c_1(TM)(\pi_2(M)) = N\mathbb{Z}$. The requirement that one of these conditions holds is equivalent to the statement:

$$A \in \pi_2(M), 2 - \dim_{\mathbb{C}} M \leq c_1(TM)(A) < 0 \implies \omega(A) \leq 0.$$

2.3. Hamiltonian dynamics. Our Hamiltonians $H = H_t \in C^\infty(M \times S^1, \mathbb{R})$ (typically time-dependent) will always be linear at infinity:

$$H = mR + \text{constant} \quad \text{for } R \gg 0$$

with slope $m \in \mathbb{R}$ not equal to a Reeb period. The Hamiltonian vector field X_H is defined by $\omega(\cdot, X_H) = dH$, and we call *1-orbits* the 1-periodic orbits x of X_H , $\dot{x}(t) = X_{H_t}(x(t))$. In the region where H is linear the 1-orbits $x(t)$ lie inside hypersurfaces $R = \text{constant}$ and correspond to the Reeb orbits $y(t) = x(t/T)$ in Σ of period $T = h'(R) < m$. The 1-orbits are the zeros of the *action 1-form*, $dA_H(x) \cdot \xi = -\int_0^1 \omega(\xi, \dot{x} - X_H) dt$ where $x \in \mathcal{L}M = C^\infty(S^1, M)$, and $\xi \in T_x \mathcal{L}M$.

2.4. A cover of the loop space.

Convention. From now on, consider only the component $\mathcal{L}_0 M \subset \mathcal{L}M$ of contractible free loops in M . We abbreviate $c_1 = c_1(TM, \omega)$.

Consider the cover of $\mathcal{L}_0 M$ introduced by Hofer-Salamon [12],

$$\widetilde{\mathcal{L}_0 M} = \{(v, x) : x \in \mathcal{L}_0 M, v : D^2 \rightarrow M \text{ a smooth disc with boundary } \partial v = x\} / \sim$$

identifying $(v_1, x_1) \sim (v_2, x_2)$ whenever $x_1 = x_2$ and ω, c_1 both vanish on the sphere $v_1 \# \overline{v_2}$ obtained by gluing the two discs together along the common boundary.

The covering group of $\widetilde{\mathcal{L}_0 M} \rightarrow \mathcal{L}_0 M$ is $\boxed{\Gamma = \pi_2(M)/\pi_2(M)_0}$ where $\pi_2(M)_0$ is generated by the spheres on which ω, c_1 both vanish. Γ acts by “gluing in spheres”. This cover is useful because the action is now well-defined:

$$A_H : \widetilde{\mathcal{L}_0 M} \rightarrow \mathbb{R}, \quad A_H(v, x) = - \int_{D^2} v^* \omega + \int_0^1 H_t(x(t)) dt.$$

2.5. \mathbb{Z} -grading on the cover. The Conley-Zehnder grading of $(v, x) \in \widetilde{\mathcal{L}_0 M}$ is well-defined and denoted $\boxed{\mu_H(v, x)}$. We refer to Salamon [23] for details.

Convention. For a C^2 -small time-independent Hamiltonian, the μ_H of a critical point of H is equal to its Morse index. Our conventions differ from [23] by reversing the sign of H , but our index μ_H in fact agrees with the μ_H of [23].

2.6. Coefficient ring Λ . The geometrical Novikov ring $\Lambda = \oplus_k \Lambda_k$ is defined using

$$\boxed{\begin{aligned} \Gamma_k &= \{ \gamma \in \Gamma : 2c_1(\gamma) = k \} && \text{(cohomological grading)} \\ \Lambda_k &= \left\{ \sum_{j=0}^{\infty} n_j \gamma_j : n_j \in \mathbb{Z}/2, \gamma_j \in \Gamma_k, \lim_{j \rightarrow \infty} \omega(\gamma_j) = \infty \right\} \end{aligned}}$$

Homological grading. in homology, the grading is reversed ($-2c_1(\gamma) = k$) so quantum cohomology/homology is compatible with Poincaré duality ([15, 11.1.16]).

Characteristic 2. We use $\mathbb{Z}/2$ to avoid the labour of discussing orientation signs, but one can do everything over characteristic 0. Also see Remark 65.

2.7. Floer cohomology. Denote

$$\mathcal{P}_k(H) = \{ c = (v, x) \in \widetilde{\mathcal{L}_0 M} : x \text{ is a (contractible) 1-orbit of } H \text{ and } \mu_H(c) = k \}$$

Then $CH^*(H, J)$ is generated over Λ by $\mathcal{P}_*(H)$:

$$CF^k(H) = \left\{ \sum_{j=0}^{\infty} n_j c_j : n_j \in \mathbb{Z}/2, c_j \in \mathcal{P}_k(H), \lim_{j \rightarrow \infty} A_H(c_j) = -\infty \right\}.$$

The choice of sign is because $A_H(\gamma \# v, x) = A_H(v, x) - \omega(\gamma)$ for $\gamma \in \Gamma$, and we want $CF^*(H) = \oplus_k CF^k(H)$ to be a Λ -module by extending the action of Γ :

$$\Gamma \ni \gamma : CF^k(H) \rightarrow CF^{k+|\gamma|}(H), \gamma \cdot (v, x) = (\gamma \# v, x)$$

where we use that $\mu_H(\gamma \# v) = \mu_H(v) + 2c_1(\gamma)$ (see [12]).

Convention. We always assume that we made a time-dependent perturbation of $(H, J) = (H_t, J_t)$ to ensure that (H, J) is regular: all 1-orbits are non-degenerate zeros of dA_H (which ensures that $CF^*(H, J)$ is finitely generated over Λ) and the following moduli spaces of Floer trajectories are smooth:

$$\begin{aligned} \mathcal{M}(x, y) &= \{ u : \mathbb{R} \times S^1 \rightarrow M : \partial_s u + J_t(\partial_t u - X_{H_t}(u)) = 0 \\ &\quad u \rightarrow x, y \text{ as } s \rightarrow -\infty, +\infty \} / (u \sim u(\cdot + \text{constant}, \cdot)) \end{aligned}$$

Separating the moduli spaces according to lifts yields dimension & energy estimates:

$$\begin{aligned} \mathcal{M}((v, x), (w, y)) &= \{ u \in \mathcal{M}(x, y) : \exists \text{ lift } \tilde{u} : \mathbb{R} \rightarrow \widetilde{\mathcal{L}_0 M} \text{ with ends } (v, x), (w, y) \} \\ \dim \mathcal{M}(c, c') &= \mu_H(c) - \mu_H(c') - 1 \\ E(u) &= \int_{\mathbb{R}} |\partial_s u|_J^2 ds = A_H(c) - A_H(c'), \quad \forall u \in \mathcal{M}(c, c') \quad (|\cdot|_J^2 = \int_{S^1} \omega(\cdot, J_t \cdot) dt) \end{aligned}$$

This energy estimate, combined with a maximum principle and a bubbling analysis, ensures that these moduli subspaces are compact up to broken trajectories. The maximum principle forces the trajectories to stay in a bounded region determined by x, y, J (using J is conical). The bubbling of J -holomorphic spheres is ruled out by the methods of Hofer-Salamon [12] (using weak monotonicity).

The differential $d : CF^k(H, J) \rightarrow CF^{k+1}(H, J)$ on a generator c' is $dc' = \sum c$ summing over $u \in \mathcal{M}(c, c')$ with $\mu_H(c) - \mu_H(c') - 1 = 0$. Extending d linearly and proving $d^2 = 0$ one obtains Floer cohomology $HF^k(H, J) = H^k(CF^*(H, J), d)$.

2.8. Continuation maps. For Hamiltonians H^\pm with slopes $m^+ \leq m^-$,

$$\varphi^* : CF^*(H^+, J^+) \rightarrow CF^*(H^-, J^-),$$

is $\varphi^*(c^+) = \sum c^-$ summing over $u \in \mathcal{N}(c^-, c^+)$ with $\mu_{H^-}(c^-) - \mu_{H^+}(c^+) = 0$, where:

$$\mathcal{C}(x^-, x^+) = \{u : \mathbb{R} \times S^1 \rightarrow M : \partial_s u + J_z(\partial_t u - X_{H_z}) = 0, \lim_{s \rightarrow \pm\infty} u(s, \cdot) = x^\pm\}$$

$$\mathcal{N}(c^-, c^+) = \{u \in \mathcal{C}(\partial c^-, \partial c^+) : \exists \text{ lift } \tilde{u} : \mathbb{R} \rightarrow \widetilde{\mathcal{L}_0 M} \text{ with ends } c^-, c^+\}$$

$$\dim \mathcal{N}(c^-, c^+) = \mu_H(c^-) - \mu_H(c^+)$$

$$E(u) = \int_{\mathbb{R}} |\partial_s u|_{J_z}^2 ds = A_H(c^-) - A_H(c^+) + \int_{\mathbb{R} \times S^1} \partial_s H_z(u) ds \wedge dt, \forall u \in \mathcal{N}(c^-, c^+)$$

where we fix some *monotone* homotopy $H_z = H_{s+it}$ from H_t^- to H_t^+ . Recall *monotone* means $H_z = h_s(R)$ for large R with $\partial_s h'_s(R) \leq 0$.

To be precise, (H_z, J_z) depend on the cylinder's coordinates $z = s + it$, with $(H_z, J_z) = (H_t^\pm, J_t^\pm)$ for large $|s|$ (so $\partial_s H_z(u) = 0$ outside a compact subset of $\mathbb{R} \times S^1$ determined by H_z), and a generic choice (H_z, J_z) ensures $\mathcal{N}(c^-, c^+)$ is smooth. The maximum principle still applies (this uses that J_z is conical at infinity, and that H_z is monotone) so u must land in a compact C determined by x^-, x^+, J_z and so in the above energy estimate $|\partial_s H_z(u)| \leq \max_C |H_z|$. This ensures $\mathcal{N}(c^-, c^+)$ are compact up to broken trajectories.

Extending φ linearly, and proving φ^* is a chain map, yields continuation maps

$$\varphi^* : HF^*(H^+, J^+) \rightarrow HF^*(H^-, J^-) \quad (m^+ \leq m^-).$$

2.9. Properties of continuation maps.

- (1) $\varphi^* : HF^*(H^+) \rightarrow HF^*(H^-)$ only depends on the slopes at infinity, since the choice (H_s, J_s) only affects the map up to a chain homotopy.
- (2) Concatenating monotone homotopies yields composites of continuation maps.
- (3) If the slopes are the same, then the continuation map is an isomorphism.

Lemma 11. *If there are no Reeb periods between m^- and m^+ , then $\varphi^* : HF^*(H^+) \rightarrow HF^*(H^-)$ is an isomorphism.*

Proof. After a continuation isomorphism which does not change the slopes at infinity, we may assume H^-, H^+ are equal except on $R \geq R_0$ where $h^- = m^- \cdot R$ and $h^+ = m(R) \cdot R$ with $m(R)$ decreasing from m^- to m^+ on a compact subinterval of $R \geq R_0$ and then remaining constantly m^+ .

All generators for H^\pm coincide and lie in the region $R < R_0$ where $H^- = H^+$. Pick a homotopy H_s from H^- to H^+ which is s -independent on $R < R_0$ and monotone on $R \geq R_0$. By the maximum principle, all continuation solutions u lie in $R < R_0$. But in that region H_s is s -independent so non-constant u would yield a 1-dimensional family: $u(\cdot + \text{constant}, \cdot)$. So the 0-dimensional moduli spaces consist of constant u 's. So $\varphi^* = \text{identity} : HF^*(H^+) \rightarrow HF^*(H^-)$. \square

2.10. Symplectic cohomology. $SH^*(M) = \varinjlim HF^*(H)$ is the direct limit over the continuation maps. It can be computed as the direct limit over a sequence of Hamiltonians with slopes $\rightarrow \infty$.

2.11. Negative slopes and Poincaré duality. Replacing incoming trajectories by outgoing trajectories defines Floer homology $HF_*(H)$:

$$CF_k(H) = \{ \prod_{j=0}^{\infty} n_j c_j : n_j \in \mathbb{Z}/2, c_j \in \mathcal{P}_k(H), \lim_{j \rightarrow \infty} A_H(c_j) = -\infty \}$$

$$\delta c = \prod c' \quad \text{taking product over all } u \in \mathcal{M}(c, c'), \mu_H(c) - \mu_H(c') - 1 = 0.$$

Lemma 12 (Poincaré duality). $CF^*(H_t) \cong CF_{2n-*}(-H_{-t})$ are canonically isomorphic chain complexes (send orbits $x(t)$ to $x(-t)$, Floer solutions $u(s, t)$ to $u(-s, -t)$).

Remark. We always deal with finitely generated modules over Λ , so CF_* , CF^* are identifiable modules, but the differentials are dual to each other. We compared $SH^*(M)$, $SH_*(M)$ in [21]. In this paper we will only use $SH^*(M)$.

2.12. Quantum cohomology and locally finite homology. The quantum cohomology as a Λ -module is $QH^*(M, \omega) = H^*(M; \Lambda)$ with underlying chain complex

$$QC^k(M) = \bigoplus_{i+j=k} C^i(M) \otimes \Lambda_j.$$

The *locally finite quantum homology* is the Λ -module

$$QH_*^{lf}(M) = H_*^{lf}(M; \Lambda).$$

Recall the latter is *locally finite homology*: at the chain level one allows infinite Λ -linear combinations of chains provided that any point of M has a neighbourhood intersecting only finitely many of the chains arising in the sum. We could identify this with a relative homology, $H_*^{lf}(M; \Lambda) \cong H_*(M; M \setminus M_0; \Lambda)$, but we will not.

By *quantum intersection product* we mean the map:

$$* : QH_*^{lf}(M) \otimes QH_*^{lf}(M) \rightarrow QH_*^{lf}(M)$$

constructed as follows. Given $\alpha \in H_i^{lf}(M)$, $\beta \in H_j^{lf}(M)$, $\gamma \in H_k(M)$, and $A \in H_2(M; \mathbb{Z})$, let $\text{GW}_{0,3,A}^M(\alpha, \beta, \gamma)$ be the genus 0 Gromov-Witten invariant (modulo 2) of J -holomorphic spheres in class A meeting generic representatives of the 1f cycles α, β and of the cycle γ . This invariant is zero for generic J unless $i + j + k = 4 \dim_{\mathbb{C}} M - 2c_1(A)$. For details, see Section 6. Then define the quantum product $*$:

$$(\alpha * \beta) \bullet \gamma = \sum_A \text{GW}_{0,3,A}^M(\alpha, \beta, \gamma) \otimes A \in \Lambda$$

where \bullet is the (ordinary) intersection product:

$$\bullet : H_*^{lf}(M) \otimes H_{2 \dim_{\mathbb{C}} M - *} (M) \rightarrow \mathbb{Z}/2$$

This determines $\alpha * \beta$, then extend Λ -linearly to $QH_*^{lf}(M)^{\otimes 2}$.

Poincaré duality is induced by ordinary Poincaré duality (see 2.6 for grading):

$$\text{PD} : QH^*(M) \cong QH_{2 \dim_{\mathbb{C}} M - *}^{lf}(M)$$

Via Poincaré duality, quantum intersection product becomes quantum cup product

$$* : QH^p(M) \otimes QH^q(M) \rightarrow QH^{p+q}(M)$$

2.13. Canonical map $c^* : QH^*(M) \rightarrow SH^*(M)$. The map c^* is the direct limit of the continuation maps $HF_*(H_0) \rightarrow HF^*(H)$, where we fix a time-independent Hamiltonian H_0 which is a C^2 -small Morse perturbation of 0, having (possibly variable) positive slopes at infinity smaller than the minimal Reeb period. The choice of H_0 does not affect $HF^*(H_0)$ or c^* by Lemma 11. For small enough H_0 , the 1-orbits of H_0 are all critical points of H_0 and the Floer trajectories are all time-independent $-\nabla H_0$ trajectories. So $CF_*(H_0) = CM_*(H_0; \Lambda)$ is the Morse complex for H_0 . Finally, Morse homology is isomorphic to ordinary homology.

Remark. Section 5.6 constructs c^* as $\psi^- : QH^*(M, \omega) \rightarrow SH^*(H)$, via a count of pseudo-holomorphic sections of a Hamiltonian fibration over a disc intersecting a given lf cycle at the disc's centre. In [21], we constructed c^* as a count of spiked discs (a $-\nabla H$ flowline from a critical point of H to the centre of a disc satisfying a Floer continuation equation). Both constructions involve the same count of discs. The spike is used to identify locally finite homology and Morse cohomology.

Now c^* intertwines the (quantum) cup product on $QH^*(M)$ and the pair-of-pants product on $SH^*(M)$: we proved this in [21] (our discussion there explains how the proof works in the non-exact setup using [18]). One only needs to prove this for $QH^*(M) \rightarrow HF^*(H_0)$ since the POP product is compatible with continuations [21]. Our proof in [21] becomes simpler now thanks to the mutually inverse

$$\boxed{\psi^- : QH^*(M) \xrightarrow{\sim} HF^*(H_0), \quad \psi^+ : HF^*(H_0) \xrightarrow{\sim} QH^*(M)}$$

which we construct in 5.6 (such ψ^\pm were the main difficulty in the proof [21]).

Lemma 13. $HF^*(H_0) \cong QH^*(M)$ is an isomorphism of rings using the pair-of-pants product and the quantum cup product, and $c^* : QH^*(M) \rightarrow SH^*(M)$ respects the product structures (in fact, also the TQFT structures).

Corollary 14. $\psi^-(1)$ is the unit for $SH^*(M)$ (see [21] for a TQFT proof).

Remark. In the Lemma, we actually compose the pair-of-pants product with a continuation map: $HF^*(H_0) \otimes HF^*(H_0) \rightarrow HF^*(2H_0) \xrightarrow{\cong} HF^*(H_0)$. By Lemma 11, for small H_0 the continuation is an identification.

Lemma 15. $HF_*(-H_0) \cong QH_*^{lf}(M)$ as Λ -algebras via Poincaré duality Lemma 12.

3. HAMILTONIAN SYMPLECTOMORPHISMS ACTION ON FLOER COHOMOLOGY

3.1. G, \tilde{G} groups, and the index $I(\tilde{g})$. Let (M, ω) be as described in Section 2.

Let $\text{Ham}(M, \omega)$ denote the group of (smooth) Hamiltonian automorphisms. Let G denote the group of (smooth) loops based at the identity:

$$G = \{g : S^1 \rightarrow \text{Ham}(M, \omega), g(0) = \text{id}\}.$$

Let K^g be the Hamiltonian generating g_t (recall any smooth path $(g_t)_{0 \leq t \leq 1}$ of Hamiltonian diffeos yields a smooth $K^g : [0, 1] \times M \rightarrow \mathbb{R}$ with $\partial_t(g_t \cdot) = X_{K^g}(t, g_t \cdot)$ and two choices of K^g differ by a constant. Since g is 1-periodic, so is K^g).

Observe that $g \in G$ acts on $\mathcal{L}_0 M \subset C^\infty(S^1, M)$ by

$$(g \cdot x)(t) = g_t(x(t)),$$

which lifts to an action on the cover $\widetilde{\mathcal{L}_0 M}$ from 2.4. Denote by \tilde{g} a choice of lift. The lifts define a group \tilde{G} [26] which is an extension of G : $1 \rightarrow \Gamma \rightarrow \tilde{G} \rightarrow G \rightarrow 1$.

We recall the Maslov index $I(\tilde{g})$ from [26]. Any $c = (v, x) \in \widetilde{\mathcal{L}_0 M}$ determines (up to homotopy) a symplectic trivialization of $x^*(TM, \omega)$, namely

$$\tau_c : x^*TM \rightarrow S^1 \times (\mathbb{R}^{2n}, \omega_0)$$

obtained by restricting a trivialization of $v^*(TM, \omega)$. A lift \tilde{g} induces (up to homotopy) a loop of symplectomorphisms $\ell(t) \in \text{Sp}(2n, \mathbb{R})$ by writing its linearization in terms of this trivialization:

$$\ell(t) = \tau_{\tilde{g}(c)}(t) \circ dg_t(x(t)) \circ \tau_c(t)^{-1}.$$

Then define the Maslov index $I(\tilde{g}) = \deg(\ell)$ where $\deg : H_1(\text{Sp}(2n, \mathbb{R})) \rightarrow \mathbb{Z}$ is the isomorphism induced by the determinant $U(n) \rightarrow S^1$ on $U(n) \subset \text{Sp}(2n, \mathbb{R})$.

$I(\tilde{g})$ is independent of the choice of (v, x) and it only depends on the homotopy class $\tilde{g} \in \pi_0(\tilde{G})$. The induced map $\pi_0(\tilde{G}) \rightarrow \mathbb{Z}$ is a homomorphism. For $g = \text{id}$ and picking \tilde{g} to be multiplication by $\gamma \in \Gamma$, $2I(\gamma) = 2c_1(\gamma)$ (homological grading).

Example 16. *If K^g is compactly supported, and we pick \tilde{g} to preserve the constants ($v \equiv x_0, x_0$) outside the support of g , then computing I for (x_0, x_0) : $I(\tilde{g}) = 0$.*

3.2. \tilde{G} -action of Floer cohomology. Define the pull-back (g^*H, g^*J) of (H, J) :

$$\boxed{g^*H_t(y) = H_t(g_t y) - K_t^g(g_t y)} \quad \boxed{J_t^g = dg_t^{-1} \circ J_t \circ dg_t}$$

The following results are just a rephrasing of the analogous result in the closed case (Seidel [26, Sec.4, Sec.5]), so we omit the proofs.

Lemma. *The pull-back of the action 1-form is $g^*(dA_H) = dA_{g^*H}$. Therefore the lift \tilde{g} induces the pull-back $\tilde{g}^*A_H = A_{g^*H} + \text{constant}$.*

Corollary 17. *The 1-orbits (being zeros of the action 1-form) biject via*

$$\text{Zeros}(dA_{g^*H}) \rightarrow \text{Zeros}(dA_H), x \mapsto g \cdot x.$$

The Floer solutions (being negative gradient trajectories of the action 1-form with respect to the metric induced by the almost complex structure), biject via

$$\begin{aligned} \mathcal{M}(c, c'; g^*H, g^*J) &\rightarrow \mathcal{M}(\tilde{g}c, \tilde{g}c'; H, J), u \mapsto g \cdot u \\ \mathcal{N}(c^-, c^+; g^*H_z, g^*J_z) &\rightarrow \mathcal{N}(\tilde{g}c^-, \tilde{g}c^+; H_z, J_z), u \mapsto g \cdot u \end{aligned}$$

where $(g \cdot u)(s, t) = g_t(u(s, t))$.

Theorem 18. *For $g \in G$ with lift \tilde{g} , we obtain an isomorphism*

$$\mathcal{S}_{\tilde{g}} : CF^*(H) \rightarrow CF^{*+2I(\tilde{g})}(g^*H), c \mapsto \tilde{g}^{-1} \cdot c$$

with $\mathcal{S}_{\tilde{g}}^{-1} = \mathcal{S}_{\tilde{g}^{-1}}$ using the reversed loop. These commute with continuations:

$$\begin{array}{ccc} CF^*(H^-, J^-) & \xrightarrow{\mathcal{S}_{\tilde{g}}} & CF^{*+2I(\tilde{g})}(g^*H^-, g^*J^-) \\ g^*\varphi \uparrow & & \uparrow \varphi \\ CF^*(H^+, J^+) & \xrightarrow{\mathcal{S}_{\tilde{g}}} & CF^{*+2I(\tilde{g})}(g^*H^+, g^*J^+) \end{array}$$

where φ is a monotone continuation map, and $g^*\varphi$ is the continuation map using

$$g^*H_z(y) = H_z(g_t y) - K_t^g(g_t y) \quad g^*J_z = dg_t^{-1} \circ J_z \circ dg_t.$$

The commutativity follows because the generators and the moduli spaces defining the continuation maps biject by Corollary 17. In particular, one can check [26, Lemma 4.1] that (g^*H, g^*J) is regular if (H, J) is, and similarly for the continuation data.

Taking direct limits, we obtain the automorphism:

$$[\mathcal{S}_{\tilde{g}}] : SH^*(M, \omega) \xrightarrow{\sim} SH^{*+2I(\tilde{g})}(M, \omega).$$

Remark. We defined $\mathcal{S}_{\tilde{g}}$ using \tilde{g}^{-1} instead of \tilde{g} because it should act by \tilde{g} on CF_* , and so on $\text{Hom}_\Lambda(CF_*, \Lambda)$ it acts by $(\tilde{g}\phi)(\cdot) = \phi(\tilde{g}^{-1}\cdot)$, but we tacitly identified $CF^* \equiv CF_*$ as Λ -modules, so we should act by \tilde{g}^{-1} on CF^* .

Remark 19. g^*H may no longer be linear at infinity and g^*J may no longer be conical. This is not a problem for Floer theory because regularity and compactness of the moduli spaces for (g^*H, g^*J) is tautologically guaranteed by that for (H, J) . The direct limit of $HF^*(g^*H, g^*J)$, where H are linear at infinity and J are conical, is isomorphic to $SH^*(M)$. This is proved by a ladder argument (like [25, Sec.4a] or like invariance in [20]) using that the minimal slope of g^*H at infinity grows to infinity as the slopes of the H grow to infinity, and using that $H_1 \leq g^*H \leq H_2$ for some Hamiltonians H_1, H_2 linear at infinity. We omit these details.

Corollary 20. $\mathcal{S}_{\tilde{g}}$ is the identity for $(g, \tilde{g}) = (\text{id}, \text{id})$, and is multiplication by γ for $(g, \tilde{g}) = (\text{id}, \gamma)$. It is a right-action: $\mathcal{S}_{\tilde{g}_1} \circ \mathcal{S}_{\tilde{g}_2} = \mathcal{S}_{\tilde{g}_2 \tilde{g}_1}$. It is homotopy invariant: if $(g_{r,t})_{0 \leq r \leq 1}$ is a smooth family of Hamiltonian automorphisms based at $g_{r,0} = \text{id}$, and $(\tilde{g}_{r,t})_{0 \leq r \leq 1}$ is a smooth lift to \tilde{G} , then on cohomology $[\varphi] \circ [\mathcal{S}_{\tilde{g}_0}] = [\mathcal{S}_{\tilde{g}_1}]$ where $\varphi : HF^*(g_0^*H, g_0^*J) \rightarrow HF^*(g_1^*H, g_1^*J)$ is the continuation isomorphism. In particular, the choice of K_t^g generating g_t does not affect the map $\mathcal{S}_{\tilde{g}}$ on cohomology.

4. CONSTRUCTION OF THE AUTOMORPHISM ON SYMPLECTIC COHOMOLOGY

4.1. Hamiltonian symplectomorphisms of linear growth. Recall $\text{Ham}_\ell(M, \omega)$ from 1.12. Let $G_\ell \subset G$ be the subgroup of all $g : S^1 \rightarrow \text{Ham}_\ell(M, \omega) \subset \text{Ham}(M, \omega)$. So g is generated by a (typically time-dependent) Hamiltonian $K_t^g : M \rightarrow \mathbb{R}$ with $m_1 R(y) + c_1 \leq K_t^g(y) \leq m_2 R(y) + c_2$, some $m_i, c_i \in \mathbb{R}$, where $R(y) \in [1, \infty)$ is the radial coordinate on the conical end. For $G_{\ell \geq 0}, G_{\ell > 0}$ require $m_1 \geq 0, m_1 > 0$.

To construct an endomorphism $\mathcal{R}_{\tilde{g}} = \varphi_H \circ \mathcal{S}_{\tilde{g}}$ of $HF^*(H)$ for a monotone continuation map φ_H , one needs $g^*H \leq H$, so we require $g \in G_{\ell \geq 0}$. To keep things simple, we will make the following simplifying assumption, although the arguments easily generalize to the results in 1.12.

Simplifying Assumption: $g \in G_\ell$ is generated by K^g of slope $\kappa > 0$ for $R \gg 0$.

This implies the Reeb flow is an S^1 -action. So, after rescaling ω , we may assume the time 1 Reeb flow is a Hamiltonian S^1 -action which is not an iterate.

Examples: S^1 -action of $g_t = e^{2\pi i t}$ on \mathbb{C}^{m+1} ; rotation in the fibres $g_t = e^{2\pi i t}$ of line bundles; S^1 -actions on Hyperkähler ALE spaces (X, ω_I) (see [20]).

4.2. Floer theoretic construction of $\mathcal{R}_{\tilde{g}}$.

Lemma. If H has slope m at infinity, then g^*H has slope $m - \kappa$.

Proof. g_t preserves R at infinity since K^g is radial there. So at infinity $g^*H = H \circ g_t - K^g \circ g_t = (m - \kappa)R + \text{constant}$. \square

Denote: $H_0 =$ generic Hamiltonian with slope $0 < \delta < (\text{min Reeb period})$
 $H_1 = H_0 + K^g$ (generic Hamiltonian with slope $\delta + \kappa$)
 $H_k = H_0 + kK^g$ (generic Hamiltonian with slope $\delta + k\kappa, k \in \mathbb{Z}$)

Then by the Lemma $g^*H_k = H_{k-1}$

In 5.6 we construct the chain maps ψ^\pm which are homotopy inverse to each other:

$$\psi^- : QC^*(M) \rightarrow CF^*(H_0) \quad \psi^+ : CF^*(H_0) \rightarrow QC^*(M).$$

To ensure ψ^+ exists we actually need H_0 to be bounded at infinity. So take H_0 a generic C^2 -small Hamiltonian, with $H_0 = h_0(R)$ convex for $R \gg 0$ and $h'_0(R) \rightarrow 0^+$ as $R \rightarrow \infty$ (H_0 is not linear at infinity, but that is not an issue). H_0, H_k should be thought of as perturbations of slope 0, $k\kappa$ Hamiltonians.

Definition. Define

$$\begin{aligned} \mathcal{R}_{\tilde{g}} &= \varphi_0 \circ \mathcal{S}_{\tilde{g}} : CF^*(H_0) \rightarrow CF^{*+2I(\tilde{g})}(H_0) \\ r_{\tilde{g}} &= \psi^+ \circ \mathcal{R}_{\tilde{g}} \circ \psi^- : QC^*(M) \rightarrow QC^{*+2I(\tilde{g})}(M) \end{aligned}$$

where φ_0 is the monotone continuation (for a homotopy from H_0 to $g^*H_0 = H_{-1}$).

$$\boxed{\begin{array}{ccccccc} QC^*(M) & \xrightarrow{\psi^-} & CF^*(H_0) & \xrightarrow{\mathcal{S}_{\tilde{g}}} & CF^{*+2I(\tilde{g})}(H_{-1}) & \xrightarrow{\varphi_0} & CF^{*+2I(\tilde{g})}(H_0) & \xrightarrow{\psi^+} & QC^{*+2I(\tilde{g})}(M) \\ & & & \searrow \mathcal{R}_{\tilde{g}} & & & \nearrow r_{\tilde{g}} & & \end{array}}$$

Theorem 21. $\mathcal{S}_{\tilde{g}}^{-k} \mathcal{R}_{\tilde{g}}^k : HF^*(H_0) \rightarrow HF^*(H_k)$ is a continuation map. For $k \geq \dim H^*(M)$, we may identify $SH^*(M) \equiv \text{image}(\mathcal{S}_{\tilde{g}}^{-k} \mathcal{R}_{\tilde{g}}^k)$ and $c^* \equiv (\mathcal{S}_{\tilde{g}}^{-k} \mathcal{R}_{\tilde{g}}^k) \circ \psi^- = (\mathcal{S}_{\tilde{g}}^{-k} \psi^-) \circ r_{\tilde{g}}^k : QH^*(M) \rightarrow SH^*(M)$. Thus $\boxed{SH^*(M) \cong QH^*(M) / \ker r_{\tilde{g}}^k}$

Proof. Abbreviate $\mathcal{S} = \mathcal{S}_{\tilde{g}}, \mathcal{R} = \mathcal{R}_{\tilde{g}}$. Consider the monotone continuation maps $\varphi_k : HF^*(H_{k-1}) \rightarrow HF^*(H_k)$. Theorem 18 yields the commutative diagram

$$\begin{array}{ccc} HF^*(H_{k+1}) & \xrightarrow[\sim]{\mathcal{S}} & HF^{*+2I(\tilde{g})}(H_k) \\ \uparrow g^* \varphi_k & & \uparrow \varphi_k \\ HF^*(H_k) & \xrightarrow[\sim]{\mathcal{S}} & HF^{*+2I(\tilde{g})}(H_{k-1}) \end{array}$$

By Property (1) in 2.9, $\varphi_{k+1} = g^* \varphi_k = \mathcal{S}^{-1} \circ \varphi_k \circ \mathcal{S}$ and so by induction, for $k \in \mathbb{Z}$,

$$\varphi_k = \mathcal{S}^{-k} \circ \varphi_0 \circ \mathcal{S}^k.$$

By property (2) in 2.9, the continuation $HF^*(H_0) \rightarrow HF^*(H_k)$ equals

$$\varphi_k \varphi_{k-1} \cdots \varphi_1 = (\mathcal{S}^{-k} \varphi_0 \mathcal{S}^k) (\mathcal{S}^{-(k-1)} \varphi_0 \mathcal{S}^{k-1}) \cdots (\mathcal{S}^{-1} \varphi_0 \mathcal{S}^1) = \mathcal{S}^{-k} (\varphi_0 \mathcal{S})^k = \mathcal{S}^{-k} \mathcal{R}^k.$$

Let $V = HF^*(H_0)$. Conjugation by \mathcal{S}^k identifies $V = HF^*(H_k)$ so φ_k becomes $\mathcal{R} : V \rightarrow V$. Using ψ^\pm we can identify $V = QH^*(M, \omega)$, which turns \mathcal{R} into $r_{\tilde{g}}$. The claims then follow by the argument in 1.3. \square

Definition. Let $\mathcal{R} : HF^*(H_\ell) \xrightarrow{\mathcal{S}} HF^{*+2I(\tilde{g})}(H_{\ell-1}) \xrightarrow{\psi} HF^{*+2I(\tilde{g})}(H_k)$ where ψ is a continuation map, so $\psi = \varphi_k \circ \cdots \varphi_\ell = \mathcal{S}^{-k} \mathcal{R}^{k-\ell+1} \mathcal{S}^{\ell-1}$ (proved like the case $\ell = 1$ above). One easily checks that these $\mathcal{R} = \mathcal{S}^{-k} \mathcal{R}^{k-\ell+1} \mathcal{S}^\ell$ form a family of maps compatible with continuation maps, so they define a map on direct limits:

$$\boxed{[\mathcal{R}_{\tilde{g}}] : SH^*(M) \rightarrow SH^{*+2I(\tilde{g})}(M)}$$

Corollary. $SH^*(M) \cong HF^*(H_0) / \ker \mathcal{R}^k$ for $k \geq \dim H^*(M)$, and $[\mathcal{R}_{\tilde{g}}] = [\mathcal{S}_{\tilde{g}}] \in \text{Aut}(SH^*(M))$. For $g_1, g_2 \in G_{\ell \geq 0}$, $[\mathcal{R}_{\tilde{g}_1}][\mathcal{R}_{\tilde{g}_2}] = [\mathcal{R}_{\tilde{g}_2 \tilde{g}_1}]$, so $[\mathcal{R}_{\tilde{g}^k}] = [\mathcal{R}_{\tilde{g}}]^k$.

Proof. The 1st claim is Theorem 21. So $\mathcal{R} : HF^*(H_0) \rightarrow HF^{*+2I(\tilde{g})}(H_0)$ induces $[\mathcal{R}]$. It is an automorphism since $\ker \mathcal{R}^{k+1} = \ker \mathcal{R}^k$. $\mathcal{S} : HF^*(H_0) \rightarrow HF^{*+2I(\tilde{g})}(H_{-1})$ determines $[\mathcal{S}]$, so $[\mathcal{S}] = [\mathcal{R}]$ since $\mathcal{S}^{-1}\mathcal{R} = \varphi_1$ represents the identity on $SH^*(M)$. The 3rd claim is Corollary 20 (or check directly on $HF^*(H_0)$). \square

4.3. Product structure on $SH^*(M)$.

Theorem 22. $\mathcal{R}_{\tilde{g}}, \mathcal{S}_{\tilde{g}}$ are compatible with products, meaning $\boxed{\mathcal{R}_{\tilde{g}}(a \cdot b) = (\mathcal{R}_{\tilde{g}}a) \cdot b}$. $\mathcal{R}_{\tilde{g}}, \mathcal{S}_{\tilde{g}}$ are pair-of-pants product by $\mathcal{R}_{\tilde{g}}(1), \mathcal{S}_{\tilde{g}}(1)$ and $r_{\tilde{g}}$ is quantum product by $r_{\tilde{g}}(1)$.

Proof. Recall [21] the product $HF^*(H_k) \otimes HF^*(H_\ell) \rightarrow HF^*(H_{k+\ell})$ counts isolated solutions $u : S \rightarrow M$ to the equation $(du - X_{H_1} \otimes \beta)^{0,1} = 0$, where S is a pair-of-pants surface (so diffeomorphic to $\mathbb{R} \times S^1 \setminus (0, 0)$) and β is a 1-form on S which equals $(k + \ell) dt, \ell dt, k dt$ near the three punctures $-\infty, (0, 0), +\infty$ and satisfies $d\beta = 0$, where a cylindrical parametrization (s, t) has been chosen near $(0, 0)$ (say $e(s, t) = (\frac{1}{4}e^{2\pi s} \cos 2\pi t, \frac{1}{4}e^{2\pi s} \sin 2\pi t)$, where $s \in (-\infty, 0]$).

This is similar to the closed setup [26, Sec.6], except we do not homotope β to zero near $s = \pm 2$ (which would contradict $d\beta = 0$, and would cause compactness problems). The only difference with the definition of product in [21] is that the Novikov ring in our current setup is larger, so we need to specify what it means for u to converge to $c^-, c_0, c^+ \in \widehat{\mathcal{L}_0 M}$. As in [26, Def.6.1], this means: if $c_0 = (v_0, x_0)$, then $u \# v_0$ (gluing v_0 onto $x_0 = \lim_{s \rightarrow -\infty} u \circ e$), viewed as a path of loops, must lift to a path in $\widehat{\mathcal{L}_0 M}$ with limits c^-, c^+ .

On homology $\mathcal{S}_{\tilde{g}}$ only depends on (g_t, \tilde{g}_t) up to homotopy by Corollary 20. So we can arrange (as in [26, Prop.6.3]) that g_t is the identity for $t \in [-\frac{1}{4}, \frac{1}{4}]$, so we can ensure $K_t^g = 0$ there. Thus near the puncture $(0, 0)$, where we use a different t coordinate than the global $t \in S^1$ of $S \cong \mathbb{R} \times S^1 \setminus (0, 0)$, the data g^*H_1, g^*J is the same as H_1, J since $K^g(t, \cdot) = 0$ there. Therefore, as in [26, Lemma 6.4], the following moduli spaces of pair-of-pants solutions biject:

$$\mathcal{M}_{(S, \beta)}(c^-, c_0, c^+; g^*H_1, g^*J) \cong \mathcal{M}_{(S, \beta)}(\tilde{g}c^-, c_0, \tilde{g}c^+; H_1, J), u(s, t) \mapsto g_t(u(s, t)).$$

We deduce $\mathcal{S}_{\tilde{g}}(a \cdot b) = (\mathcal{S}_{\tilde{g}}a) \cdot b$ for any $g \in G$. Since continuation maps preserve the product structure [21], the same holds for $\mathcal{R}_{\tilde{g}}$ (using the Remark after Lemma 13). So, using the unit $1 = \psi^-(1)$ of Corollary 14: $\mathcal{R}_{\tilde{g}}(a) = \mathcal{R}_{\tilde{g}}(1) \cdot a$, $\mathcal{S}_{\tilde{g}}(a) = \mathcal{S}_{\tilde{g}}(1) \cdot a$.

Recall $r_{\tilde{g}} = \psi^+ \mathcal{R}_{\tilde{g}} \psi^-$. So $r_{\tilde{g}}(y) = \psi^+[\mathcal{R}_{\tilde{g}}(1) \cdot \psi^-(y)]$. By Corollary 14: $\mathcal{R}_{\tilde{g}}(1) = \psi^- r_{\tilde{g}}(1)$. By Lemma 13: $\psi^+[\psi^-(r_{\tilde{g}}(1)) \cdot \psi^-(y)] = r_{\tilde{g}}(1) * y$, using $\psi^+ \psi^- = \text{id}$. \square

Abbreviate $c = r_{\tilde{g}}(1)$. Identifying $QH^*(M) \equiv HF^*(H_k)$ the continuation $\varphi^k : HF^*(H_0) \rightarrow HF^*(H_k)$ is quantum cup product by c_Q^k (quantum powers). By Lemma 13, $\alpha_0 * \beta_0 = \alpha_0 \cdot \beta_0$ via $QH^*(M) \equiv HF^*(H_0)$, but not on $HF^*(H_k)$ unless one correctly interprets k . For general reasons [21] \cdot will not preserve H_k :

$$HF^*(H_k) \otimes HF^*(H_\ell) \rightarrow HF^*(H_{k+\ell}), \alpha_k \otimes \beta_\ell \mapsto \alpha_k \cdot \beta_\ell.$$

This can be elucidated in our case. Suppose α_k, β_ℓ have $HF(H_0)$ representatives: $\alpha_k = \varphi^k(\alpha_0), \beta_\ell = \varphi^\ell(\beta_0)$. Since continuations are compatible with products [21],

$$\varphi^{k+\ell}(\alpha_0 * \beta_0) = \varphi^{k+\ell}(\alpha_0 \cdot \beta_0) = \varphi^k(\alpha_0) \cdot \varphi^\ell(\beta_0) = \alpha_k \cdot \beta_\ell,$$

which proves that $\alpha_k \cdot \beta_\ell = c_Q^{k+\ell} * \alpha_0 * \beta_0 = (c_Q^k * \alpha_0) * (c_Q^\ell * \beta_0) = \alpha_k * \beta_\ell$.

5. PSEUDOHOLOMORPHIC SECTIONS

5.1. Space of sections $\mathcal{S}(j, \hat{J})$. We briefly recall some definitions [Sec. 7, [26]].

Let $(\pi : E \rightarrow S^2, \Omega)$ be a *symplectic fibre bundle* with fibre (M, ω) , meaning: Ω_z is a symplectic form for the fibre E_z over $z \in S^2$, smoothly varying in z . It is understood that we fix an isomorphism $i : (M, \omega) \rightarrow (E_{z_0}, \Omega_{z_0})$ where $z_0 \in S^2$ is the South pole (view $S^2 = D^+ \cup_{S^1} D^-$ as a union of two discs, $z_0 = \text{centre}(D^-)$).

Let $\mathcal{J}(E, \Omega)$ be the space of $J = (J_z)_{z \in S^2}$ (smooth in z), where J_z is a conical structure on the fibre (E_z, Ω_z) (see 2.1). Fix a positively oriented complex structure j on S^2 . Call an almost complex structure \hat{J} on E *compatible* with (j, J) if $d\pi \circ \hat{J} = j \circ d\pi$ and \hat{J} restricts to J fibrewise. Denote $\hat{\mathcal{J}}(j, J)$ the space of compatible \hat{J} .

Definition. For $j, J, \hat{J} \in \hat{\mathcal{J}}(j, J)$ as above, denote $\mathcal{S}(j, \hat{J})$ the space of (j, \hat{J}) -holomorphic sections, meaning all $s : S^2 \rightarrow E$ with

$$ds \circ j = \hat{J} \circ ds.$$

Definition. Call (E, Ω) *Hamiltonian (symplectic fibre bundle)* if there is a closed two-form $\tilde{\Omega}$ on E restricting to Ω_z fibrewise.

Definition. For $(E, \Omega, \tilde{\Omega})$ Hamiltonian, two sections s, s' are Γ -equivalent if $\tilde{\Omega}(s) = \tilde{\Omega}(s')$ and $c_1(TE^v, \Omega)(s) = c_1(TE^v, \Omega)(s')$, where $TE^v = \ker d\pi$. Denote $\mathcal{S}(j, \hat{J}, S) \subset \mathcal{S}(j, \hat{J})$ the subspace of sections in the Γ -equivalence class S .

The equivalence classes do not depend on the choice of $\tilde{\Omega}$, but only on Ω : a difference of two sections $S - S' \in \pi_2(E)$ maps to $[S^2] - [S^2] = 0 \in \pi_2(S^2)$ via the fibration $E \rightarrow S^2$, so $S - S' \in \text{im}(\pi_2(M) \rightarrow \pi_2(E))$, so the Ω -value on this fibre class determines whether $\tilde{\Omega}(S - S')$ is zero or not.

By [26, Lemma 2.10] for S, S' , there is a unique $\gamma \in \Gamma$ such that the $\tilde{\Omega}$ values on S, S' differ by $\omega(\gamma)$, and the $c_1(TE^v, \Omega)$ values differ by $c_1(TM, \omega)(\gamma)$. Conversely, given $S, \gamma \in \Gamma$ there is a unique class denoted $S' = S + \gamma$ for which this holds.

5.2. Hamiltonian fibration. From now on, we view S^2 as $D^+ \cup_{S^1} D^-$ and we will use the coordinates $z = \exp(s + it)$ on $D^+ = \{z \in \mathbb{C} : |z| \leq 1\}$ and the complex structure $j\partial_s = \partial_t$. Near ∂D^+ these coordinates lie in $(s, t) \in (-\epsilon, 0] \times S^1$ and we can extend these coordinates to D^- near ∂D^- via $(s, t) \in [0, \epsilon) \times S^1$.

Definition. Given $g \in \text{Ham}_\ell(M, \omega)$ generated by $(K_t^g)_{t \in S^1}$, define the symplectic fibre bundle $(\pi_g : E_g \rightarrow S^2, \Omega_g)$ by the clutching construction

$$\begin{aligned} E_g &= (D^+ \times M) \cup_{\phi^g} (D^- \times M) \\ \phi^g : \partial D^+ \times M &\rightarrow \partial D^- \times M, \phi^g(t, y) = (t, g_t(y)) \end{aligned}$$

with form $\Omega_g = \omega^\pm$ on the fibres (the pull-backs of ω from M to $D^\pm \times M$).

Let $H^\pm : D^\pm \times M \rightarrow \mathbb{R}$ be Hamiltonians which:

- (1) vanish near the centres of D^\pm ;
- (2) only depend on $t \in S^1$ near ∂D^\pm : $H^\pm(s + it, y) = H_t^\pm(y)$;
- (3) and which satisfy the gluing condition on $\partial D^\pm \times M$:

$$H_t^+(y) = g^* H_t^-(y) = H_t^-(g_t y) - K_t^g(g_t y).$$

Definition. We call H monotone if $\partial_s(h^+)' \leq 0$ and $\partial_s(h^-)' \leq 0$, where $H^\pm = h^\pm(R)$ at infinity and where s is the coordinate determined by the parametrizations

$$\begin{aligned} (-\infty, 0] \times S^1 &\rightarrow D^+ \setminus z_\infty \equiv \{z \in \mathbb{C} \setminus 0 : |z| \leq 1\}, & (s, t) &\mapsto e^{2\pi(s+it)} \\ [0, +\infty) \times S^1 &\rightarrow D^- \setminus z_0 \equiv \{z \in \mathbb{C} : |z| \geq 1\}, & (s, t) &\mapsto e^{2\pi(s+it)}. \end{aligned}$$

Define a one-form τ^\pm on $D^\pm \times M$ and a closed 2-form $\tilde{\Omega}$ on E_g by

$$\begin{aligned} \tau^\pm &= H^\pm dt \\ \tilde{\Omega}|_{D^\pm \times M} &= \omega^\pm - d\tau^\pm = \omega^\pm - dH^\pm \wedge dt - \partial_s H^\pm ds \wedge dt. \end{aligned}$$

Note $\tilde{\Omega}$ glues correctly since $g_t^* \omega = \omega$ and since

$$((\phi^g)^* \omega^-)(\vec{v}, \partial_t) = \omega^-(d\phi^g \cdot \vec{v}, X_{K_g} \circ g_t) = d(K_t^g \circ g_t dt) \cdot (\vec{v}, \partial_t).$$

Definition 23. Recall [26, Lemma 2.12], that E_g has a section $s_{\tilde{g}}$ built as follows. Pick any $c \in \widetilde{\mathcal{LM}}$, and pick representatives $(v, x), (v', x')$ of $c, \tilde{g}(c)$. Then glue:

$$\begin{aligned} s_{\tilde{g}}^+ : D^+ &\rightarrow D^+ \times M, s_{\tilde{g}}^+(z) = (z, v(z)) \\ s_{\tilde{g}}^- : D^- &\rightarrow D^- \times M, s_{\tilde{g}}^-(z) = (z, \overline{v'}(z)) \end{aligned}$$

where $\overline{v'} : D^- \cong D^2 \rightarrow M$ involves an orientation-reversing identification \cong .

The Γ -equivalence class $S_{\tilde{g}}$ of $s_{\tilde{g}}$ does not depend on the choices c, v, v' , and

$$I(\tilde{g}) = -c_1(TE_g^v, \Omega_g)(s_{\tilde{g}}).$$

5.3. Admissible almost complex structures \hat{J} .

Remark. To build a symplectic form on the total space of a symplectic fibration (Thurston's method) one modifies the symplectic form by a pull-back of a large multiple of a symplectic form on the base to achieve non-degeneracy in the horizontal distribution. This fails in our case because the fibres are non-compact and the given symplectic form grows like R at infinity, so such pull-backs cannot dominate. Thus [26, Lemma 7.4] fails in our setup. The remedy is to require that \hat{J} has a special form at infinity, depending on the Hamiltonian.

Definition 24. Call (j, J, \hat{J}) admissible if J_z is $(\Omega_g)_z$ -compatible, $\hat{J} \in \hat{\mathcal{J}}(j, J)$ and such that for large R they have the form

$$\hat{J}_{(z,y)} = \begin{pmatrix} j & 0 \\ \nu_{(z,y)} \circ j & J_z \end{pmatrix} = \begin{pmatrix} j & 0 \\ ds \otimes X_H - dt \otimes J_z X_H & J_z \end{pmatrix}$$

where $\nu_{(z,y)} : T_z S^2 \rightarrow T_y M$ is the (j, J_z) -antilinear homomorphism given by

$$\nu_{(z,y)} = ds \otimes J_z X_H(z, y) + dt \otimes X_H(z, y).$$

Remark 25. (due to Gromov) The $\hat{J}_z = \begin{pmatrix} j & 0 \\ \nu_{(z,y)} \circ j & J_z \end{pmatrix}$ arise from turning the Floer continuation $\partial_s u + J_z(u)(\partial_t u - X_H(z, u)) = 0$, for $z = (s, t) \in \mathbb{R} \times S^1$, into

$$du + J \circ du \circ j = \nu,$$

and finally into $ds \circ j = \hat{J} \circ ds$ for $s : \mathbb{R} \times S^1 \rightarrow \mathbb{R} \times S^1 \times M$, $s(z) = (z, u(z))$.

Remark. Our H, \hat{J} correspond in the notation of [15, Sec.8.1 (p.243)] to G and $\tilde{J}_G dt = \tilde{J}_\tau$ (their Hamiltonian vector fields are opposite to ours). The curvature [15, Sec.8.1] is $F_\tau \text{vol}_{S^2} = \partial_s H \text{vol}_{S^2}$. The $\tilde{\Omega}$ -horizontal distribution over D^+ is $\text{Hor} = \{\xi \in T(D^+ \times M) : \tilde{\Omega}(\ker d\pi, \xi) = 0\} = \text{span}\{\partial_s, \partial_t + X_H\}$. \hat{J} preserves Hor .

Lemma 26. *For admissible (j, J, \hat{J}) , if $\partial_s H \leq 0$ then for large $c \in \mathbb{R}$, \hat{J} is $\tilde{\Omega} + \pi^*(c \cdot \text{vol}_{S^2})$ -compatible and so $\tilde{\Omega} + \pi^*(c \cdot \text{vol}_{S^2})$ is symplectic. Without the condition $\partial_s H \leq 0$, this still holds provided we assume $\partial_s H$ is bounded above.*

Proof. At infinity, a computation shows that:

$$\begin{aligned} \tilde{\Omega}(a\partial_s + b\partial_t + \vec{m}, a'\partial_s + b'\partial_t + \vec{m}') &= \omega(\vec{m} - bX_H, \vec{m}' - b'X_H) - (ab' - a'b)\partial_s H \\ \tilde{\Omega}(a\partial_s + b\partial_t + \vec{m}, \hat{J}(a\partial_s + b\partial_t + \vec{m})) &= \omega(\vec{m} - bX_H, J(\vec{m} - bX_H)) - (a^2 + b^2)\partial_s H \end{aligned}$$

adding $\pi^*(c \cdot ds \wedge dt)$ to $\tilde{\Omega}$ contributes an extra $c \cdot (a^2 + b^2)$. Since J is ω -compatible, this proves the claim at infinity for $c \geq \partial_s H$.

In the compact region where $\hat{J} = \begin{pmatrix} j & 0 \\ \nu \circ j & J \end{pmatrix}$ does not have ν in the special form of Definition 24, we need positivity of:

$$\omega(\vec{m}, J\vec{m}) + \omega(\vec{m} - bX_H, \nu j(a\partial_s + b\partial_t)) - \omega(bX_H, J\vec{m}) - \omega(\vec{m}, aX_H) + (a^2 + b^2)(c - \partial_s H).$$

Abbreviate $|\vec{m}|^2 = \omega(\vec{m}, J\vec{m})$. By rescaling, assume $a^2 + b^2 + |\vec{m}|^2 = 1$. If $a^2 + b^2 \ll |\vec{m}|^2$, then the first term dominates (on the compact region all terms are bounded). Otherwise, we make the last term dominate by making $c \gg 0$. \square

Example. Let $H_0 = \delta(R)R + \text{constant}$ for $R \gg 0$, with bounded concave $\delta(R)R > 0$. By Lemma 11, $HF^*(H_0)$ is the same as if $\delta(R) < (\text{min Reeb period})$ was constant. The advantage: the non-monotone homotopy H_s from 0 to H_0 has $\partial_s H_s$ bounded.

Lemma 27. *Suppose $\partial_s H \leq 0$. For any (j, \hat{J}) -holomorphic section $u : S^2 \rightarrow E_g$, $u^* \tilde{\Omega} \geq 0$ at all points z for which $u(z)$ lies in the region where \hat{J} has the special form as in Definition 24.*

Proof. Locally $u(z) = (z, u^\pm) \in D^\pm \times M$. Using the proof of Lemma 26:

$$\begin{aligned} \frac{u^* \tilde{\Omega}}{ds \wedge dt} &= \tilde{\Omega}(du \circ \partial_s, du \circ j\partial_s) = \tilde{\Omega}(\partial_s u, \hat{J}\partial_s u) \\ &= \tilde{\Omega}(1 \cdot \partial_s + \partial_s u^\pm, \hat{J}(1 \cdot \partial_s + \partial_s u^\pm)) = \omega(\partial_s u^\pm, J\partial_s u^\pm) - \partial_s H \geq 0. \quad \square \end{aligned}$$

5.4. Compactness result for $\mathcal{S}(j, \hat{J})$. By Lemma 26, $\tilde{\Omega} + \pi^*(\sigma)$ is symplectic on $E = E_g$ for some form σ on S^2 , and admissible \hat{J} are $\tilde{\Omega} + \pi^*(\sigma)$ -compatible.

Lemma 28. *Under the assumptions of Lemma 26, and J generic, then for every $C \in \mathbb{R}$, and any given compact $D \subset E_{z_0}$, only finitely many Γ -equivalence classes S have $\tilde{\Omega}(S) \leq C$ with $\mathcal{S}(j, \hat{J}, S)$ containing a section intersecting D over z_0 .*

Proof. Consider a sequence $s_n \in \mathcal{S}(j, \hat{J}, S)$ with $\tilde{\Omega}(s_n) \leq C$, $c_1(TE^v, \Omega)(s_n) \leq c$ and with $s_n(z_0) \rightarrow y \in E_{z_0}$.

Three out of four possible failures of sequential compactness are analogous to the case of closed manifolds [26, Lemmas 7.5, 7.6]. These three failures would imply the existence of a holomorphic section $s \in \mathcal{S}(j, \hat{J})$, which respectively: (1) passes through y but $c_1(TE^v, \Omega)(s) < c$; or (2) passes through y and $c_1(TE^v, \Omega)(s) = c$ but a holomorphic bubble appears in some fibre E_z and intersects $s(z)$; or (3) a cusp-curve of total Chern number $\leq c - c_1(TE^v, \Omega)(s)$ appears in E_{z_0} whose initial marked point lands at $s(z_0)$ and whose last marked point lands at y .

Failure (4): the s_n are unbounded in the fibre direction. This cannot happen by Lemma 30. \square

Lemma 29 (Monotonicity Lemma). *Suppose $\partial_s H$ is bounded above and (j, J, \hat{J}) admissible. Then there is a constant $C > 0$, such that for any (j, \hat{J}) pseudo-holomorphic disc $s : D \subset S^2 \rightarrow E_g$ and boundary $s(\partial D)$ lying in the boundary of a ball of radius ϵ with centre intersecting $s(D)$, the energy $E(s) = \int_D \|du\|_j^2 ds \wedge dt$ calculated with respect to the metric $(\tilde{\Omega} + \pi^* \sigma)(\cdot, \hat{J} \cdot)$ (Lemma 26) is at least $C\epsilon^2$.*

Proof. This is a standard consequence of the isoperimetric inequality, see [1, Sec.4.3]. This uses the fact that M , and hence E_g , is geometrically bounded. In particular, $X_H = h'_z(R)\mathcal{R}$ is bounded at infinity since the slope $h'_z(R)$ is bounded and \hat{J} is prescribed in terms of j, J, X_H at infinity by Definition 24. The condition that $\partial_s H$ is bounded above is required for Lemma 26 to hold. \square

Lemma 30. *Suppose $\partial_s H$ is bounded above and (j, J, \hat{J}) admissible. Then all sections $s \in \mathcal{S}(j, \hat{J})$ which intersect a compact domain $D \subset E_{z_0}$ with $\tilde{\Omega}(s) \leq C$ are contained in a compact region of E_g determined by C, D .*

Proof. Follows by Lemma 29, and the energy estimate $(\tilde{\Omega} + \pi^* \sigma)(s) \leq C + \sigma[S^2]$. \square

Lemma 31 (Maximum principle). *Assume that for $R \geq R_0$ the following hold: $H = h_z(R)$, $\partial_s h'_z \leq 0$ and \hat{J} has the form as in Definition 24. Then all (j, \hat{J}) pseudo-holomorphic sections $s : S^2 \rightarrow E_g$ lie in the region $R \leq R_0$.*

Proof. By Remark 25, s has the form $u : \mathbb{R} \times S^1 \rightarrow \mathbb{R} \times S^1 \times M$ (defined on a subset of $\mathbb{R} \times S^1$), with $du \circ j = J \circ du + \nu \circ j$. Let $\rho = R \circ u$. Since $dR \circ J = -\theta = -R\alpha$,

$$d\rho \circ j = dR(J \circ du + \nu \circ j) = -u^* \theta + dt \otimes \theta(X_H)$$

using $dR(X_H) = dR(h'(R)\mathcal{R}) = 0$. Arguing as for the Maximum Principle in [21],

$$\Delta \rho ds \wedge dt = -d(d\rho \circ j) = \frac{1}{2} \|du - X_H \otimes dt\|^2 + \frac{h' d\rho \wedge dt - d(\rho h' dt)}{ds \wedge dt}$$

so $(\Delta \rho + \text{first order terms in } \rho) \geq -\rho(\partial_s h')$. So the maximum principle for ρ applies provided $\partial_s h' \leq 0$. \square

5.5. Transversality for $\mathcal{S}(j, \hat{J})$.

Lemma 32. *After a small generic perturbation of (J, H) , for admissible (j, J, \hat{J}) the moduli space $\mathcal{S}(j, \hat{J})$ is a smooth manifold of dimension*

$$d(s) = (\dim_{\mathbb{R}} \text{ of } \mathcal{S}(j, \hat{J}) \text{ near } s) = 2 \dim_{\mathbb{C}} M + 2c_1(TE_g^v, \Omega)(s),$$

where $TE^v = \ker d\pi$ (abbreviating $E = E_g$), and the evaluation maps

$$\begin{aligned} \text{ev} : S^2 \times \mathcal{S}(j, \hat{J}) &\rightarrow E, \text{ ev}(z, s) = s(z) \\ \text{ev}_{z_0} : \mathcal{S}(j, \hat{J}) &\rightarrow M, \text{ ev}_{z_0}(z, s) = i^{-1}(s(z_0)) \end{aligned}$$

are transverse respectively to

$$\begin{aligned} \eta : \mathcal{M}_0^s(J) \times_{PSL(2, \mathbb{C})} \mathbb{CP}^1 &\rightarrow E, \eta(z, w, x) = w(x) \\ \eta_1 : \mathcal{C}_{r, k}(J) &\rightarrow M, \eta_1(w_1, \dots, w_r, t_1, \dots, t_r, t'_1, \dots, t'_r) = w_1(t'_1), \end{aligned}$$

so the evaluation $\text{ev}_{z_0} : \mathcal{S}(j, \hat{J}, S) \rightarrow M$ is a pseudo-cycle of dimension $d(S)$.

Notation: $\mathcal{M}_k^s(\mathbf{J}) = \{(z, w) \in S^2 \times C^\infty(\mathbb{CP}^1, E) : w \text{ simple } J_z\text{-holomorphic curve in } E_z \text{ with } c_1(TE_z, \Omega_z)(w) = k\}$. This is empty for $k < 0$ and is a $\dim E$ -manifold for $k = 0$ whose image under η has $\text{codim} = 4$ (uses weak^+ -monotonicity).

$\mathcal{C}_{\mathbf{r},\mathbf{k}}(\mathbf{J})$ is the $(2n + 2k - 2r)$ -manifold of simple J -holomorphic cusp-curves with $r \geq 1$ components of total Chern number k quotiented by the $PSL(2, \mathbb{C})^r$ action, where $w_i(t_i) = w_{i+1}(t'_{i+1})$ are the nodes for $i = 1, \dots, r - 1$.

This Lemma is the analogue of [26, Prop.7.3], except at infinity we perturb \hat{J} in a controlled way by perturbing H (thus preserving admissibility). The proof of transversality using perturbations of H is in Sec. 8.3 & 8.4 of [15]. The proof that ev_{z_0} is a pseudo-cycle then follows by Lemma 28, just like in [26, Prop.7.7]. Indeed, the proof of Lemma 28 describes how $ev_{z_0}(\mathcal{S}(j, \hat{J}, S))$ can be compactified by countably many images of manifolds (since we only care about the image, we may assume the holomorphic bubbles and cusp-curves that we described are simple). By a dimension count, using the above transverseness claims about η, η_1 , one shows that these additional manifolds have dimension $\leq d(S) - 2$.

Remark 33. $\mathcal{S}(j, \hat{J}, S)$ depends on H in so far as \hat{J} depends on H (admissibility), but equivalence classes S are independent of this choice by 5.1 (they depend on Ω_g).

5.6. Construction of the ψ^+ and ψ^- maps.

Theorem 34. Let H_0 be a Hamiltonian on M which at infinity equals $h(R)$ (non-linear) with slopes $0 < h' < (\min \text{Reeb period})$, $h'' < 0$ and $h' \rightarrow 0$ fast enough so that H_0 is bounded. Then there are chain maps

$$\psi^+ : CF^*(H_0) \rightarrow QC_{2n-*}^{lf}(M) \quad \psi^- : QC_{2n-*}^{lf}(M) \rightarrow CF^*(H_0)$$

homotopy inverse to each other, where $2n = \dim_{\mathbb{R}} M$. Via Poincaré duality:

$$\psi^+ : CF^*(H_0) \rightarrow QC^*(M) \quad \psi^- : QC^*(M) \rightarrow CF^*(H_0)$$

Proof. ψ^+ will count $(j, \hat{J}^+ = \hat{J}|_{D^+})$ -holomorphic sections $s^+ : D^+ \rightarrow D^+ \times M$, for (j, J, \hat{J}) admissible, where $H^+ = H_0$ on ∂D^+ and $H^+ = 0$ at the centre of the disc (compare 5.2). We can ensure that $\partial_s H_z$ is bounded above since H_0 is bounded.

Let $c = (v, x) \in \widetilde{\mathcal{L}_0 M}$, where x is a 1-orbit of X_{H_0} . Denote $\mathcal{M}^+ = \mathcal{M}^+(c; H^+, \hat{J}^+)$ the moduli space of such sections s^+ with $(D^2 \cong D^+ \xrightarrow{s^+} M) = c \in \widetilde{\mathcal{L}_0 M}$, where \cong is the orientation-preserving identification. These moduli spaces are defined in the closed setup in [26, Sec.8]. For generic (\hat{J}, H) , \mathcal{M}^+ is smooth and

$$\dim \mathcal{M}^+(c; H^+, \hat{J}^+) = 2n - \mu_{H_0}(c),$$

(see 2.5 for gradings). The evaluation at the centre of the disc $ev_{z_\infty} : \mathcal{M}^+ \rightarrow M, u \mapsto u(z_\infty)$ is a locally finite pseudo-cycle of that dimension. To ensure the locally finite condition, we use Lemma 29 and the a priori energy estimate for u :

$$\begin{aligned} E(u) &= \int_{D^+} \|u\|_{(\tilde{\Omega} + \pi^* \sigma)(\cdot, \hat{J}^+)}^2 ds \wedge dt \\ &= (\tilde{\Omega} + \pi^* \sigma)[c] \\ &= \int_{D^+} u^* \omega + \int_{D^+} u^* d(-H^+ dt) + \int_{D^+} u^* (\pi^* \sigma) \\ &= \omega[c] - \int_{S^1} H^+(x) dt + \sigma[D^+]. \end{aligned}$$

Indeed, this estimate and Lemma 29 imply that all $u \in \mathcal{M}^+(c; H^+, \hat{J}^+)$ which intersect a given compact C' of M must lie in a compact subset C'' of M determined by C' . But then a standard Gromov compactness argument implies the compactness up to breaking of the subset of all $u \in \mathcal{M}^+(c; H^+, \hat{J}^+)$ intersecting C' .

Define ψ^+ by extending linearly the map defined on generators by

$$\begin{aligned} \psi^+ : CF^*(H_0) &\rightarrow QC_{2n-*}^{lf}(M), \\ \psi^+(c) &= \sum_{\gamma \in \Gamma} [\text{ev}_{z_\infty}(\mathcal{M}^+(\gamma \cdot c; H^+, \hat{J}^+))] \otimes \langle \gamma \rangle \end{aligned}$$

As $\dim \mathcal{M}^+(\gamma \cdot c; H^+, \hat{J}^+) = 2n - |\gamma \cdot c| = 2n - |c| - |\gamma|$, the right hand side above has degree $2n - |c|$. For $c = (v, x)$, the energy of $u \in \mathcal{M}^+(\gamma \cdot c; H^+, \hat{J}^+)$ is

$$E(u) = \omega(\gamma) + \omega[c] - \int_{S^1} H^+(x) dt + \sigma[D^+]$$

So for fixed c but varying γ , the $\omega(\gamma)$ must grow to ∞ if such energies were to grow to ∞ . So ψ^+ is well-defined.

Similarly define $\mathcal{M}^-(c; H^-, \hat{J}^-)$ requiring $(D^2 \cong D^- \xrightarrow{s^-} M) = c \in \widetilde{\mathcal{L}_0 M}$, where \cong is orientation-reversing. Then $\dim \mathcal{M}^-(c; H^-, \hat{J}^-) = \mu_{H_0}(c)$. Since H^- is a homotopy from H_0 to 0 we can choose it to be monotone: $\partial_s H_z^- \leq 0$. So by Lemma 31 we obtain a (finite) pseudo-cycle $\text{ev}_{z_0} : \mathcal{M}^-(c; H^-, \hat{J}^-) \rightarrow M$.

$$\begin{aligned} \psi^- : QC_{2n-*}^{lf}(M) &\rightarrow CF^*(H_0), \\ \psi^-(\alpha) &= \sum_{\dim \mathcal{M}^-(c; H^-, \hat{J}^-) + \dim \alpha = 2n} ([\text{ev}_{z_0}(\mathcal{M}^-(c; H^-, \hat{J}^-))] \bullet \alpha) \langle c \rangle \end{aligned}$$

where \bullet is the intersection product between the pseudo-cycle ev_{z_0} and the lf cycle α . In particular, for the unit $[M] \otimes 1 \in QH_{2n}^{lf}(M)$,

$$\psi^-([M]) = \sum_{\mu_{H_0}(c)=2n} \# \mathcal{M}^-(c; H^-, \hat{J}^-) \cdot \langle c \rangle.$$

By standard arguments (combining [18] and [21]), one checks that ψ^-, ψ^+ are chain maps inverse to each other up to chain homotopy. We omit the details. \square

5.7. Algebro-geometric construction of $\mathbf{r}_{\tilde{g}}$.

Theorem 35. $r_{\tilde{g}}(1) \in QH^{2I(\tilde{g})}(M)$ is represented Poincaré dually by the lf cycle

$$r_{\tilde{g}}[M] = \sum_{\gamma \in \Gamma} [\text{ev}_{z_\infty}(\mathcal{S}(j, \hat{J}, \gamma + S_{\tilde{g}}))] \otimes \gamma \in QC_{2n-2I(\tilde{g})}^{lf}(M).$$

After Poincaré dualizing $r_{\tilde{g}}$, and for a generic lf chain $\alpha : \Delta^{|\alpha|} \rightarrow M$,

$$\begin{aligned} r_{\tilde{g}} : QH_{2n-*}^{lf}(M) &\rightarrow QH_{2n-*+2I(\tilde{g})}^{lf}(M) \\ r_{\tilde{g}}(\alpha \otimes 1) &= \sum_{\gamma \in \Gamma} \left[\text{ev}_{z_\infty} \left(\mathcal{S}(j, \hat{J}, \gamma + S_{\tilde{g}}) \times_{\text{ev}_{z_0}, \alpha} \Delta^{|\alpha|} \right) \right] \otimes \gamma \\ &= \sum_{\gamma \in \Gamma} \sum_i \left[(\text{ev}_{z_\infty} \times \text{ev}_{z_0}) \left(\mathcal{S}(j, \hat{J}, \gamma + S_{\tilde{g}}) \right) \right] \bullet [D[\beta_i] \times \alpha] \beta_i \otimes \gamma \end{aligned}$$

counts holomorphic sections intersecting the lf chain α over z_0 and the (finite) chain $D[\beta_i]$ over z_∞ , where $D[\beta_i]$ is the dual basis with respect to the intersection product $\bullet : H_*^{lf}(M) \otimes H_*(M) \rightarrow \mathbb{Z}/2$ of a basis β_i of lf cycles for $H_*^{lf}(M)$.

Proof. Recall from Lemma 15 that

$$QH^*(M) \cong HF^*(H_0) \cong HF_{2n-*}(-H_0) \cong QH_{2n-*}^{lf}(M).$$

So $\psi^- : QH_{2n-*}^{lf}(M) \rightarrow HF^*(H_0)$ factors through $HF_{2n-*}(-H_0)$ (canonically identified with $HF^*(H_0)$ by identifying generators), and the intermediate map $QH_{2n-*}^{lf}(M) \rightarrow HF_{2n-*}(-H_0)$ equals the Ψ^+ map of [26] for the data $-H_{-(s+it)}$ on D^+ (which is dual to the data H_{s+it} on D^- by Lemma 12).

Similarly, our composite $r_{\tilde{g}} = \psi^+ \circ \varphi_0 \circ \mathcal{S}_{\tilde{g}} \circ \psi^-$ is analogous to the composite $\Psi^- \circ \varphi_0 \circ HF_*(\tilde{g}^{-1}) \circ \Psi^+$ which arises in [26, Sec.8] but using the dual data $-H_{-(s+it)}$ instead of H_{s+it} . The gluing argument of [26, Sec.8] proves that the image of the unit $[M] \otimes 1 \in QH_{2n}^{lf}(M)$ under $r_{\tilde{g}} : QH_{2n-*}^{lf}(M) \rightarrow QH_{2n-*}^{lf}(-2I(\tilde{g}))(M)$ is

$$r_{\tilde{g}}([M]) = \sum_{\gamma \in \Gamma} [\text{ev}_{z_\infty}(\mathcal{S}(j, \hat{J}, \gamma + S_{\tilde{g}}))] \otimes \gamma \in QH_{2n-2I(\tilde{g})}(M)$$

(we evaluate at z_∞ instead of z_0 because of the dualization which changes domain coordinates). In particular, since gluing sections s^+, s^- representing $c', \tilde{g}c'$ defines the equivalence class $S_{\tilde{g}}$ (for any c'), the gluing of $s^+ \in \mathcal{M}^+(\gamma \cdot \tilde{g}^{-1}c; H^+, \hat{J}^+)$ and $s^- \in \mathcal{M}^-(c; H^-, \hat{J}^-)$ yields a section of E_g in the class $\gamma + S_{\tilde{g}}$ (take $c' = \tilde{g}^{-1}c$).

The same gluing argument (since we are only changing the intersection conditions over z_0, z_∞) in fact shows more generally that $r_{\tilde{g}} = \psi^+ \circ \varphi_0 \circ \mathcal{S}_{\tilde{g}} \circ \psi^-$ agrees on homology with the map in the claim. \square

Remark 36. *The map $r_{\tilde{g}}$ is not in general an isomorphism, unlike for closed M . This is because the inverse map can no longer be defined: it would involve a non-monotone homotopy H_s from H_{-1} to H_0 which has $\partial_s H_s$ unbounded above.*

5.8. Invariance: the choice of \hat{J} . In Theorem 35 we did not specify precisely the choice of \hat{J} : the proof recovers \hat{J} as a gluing of admissible \hat{J} over the discs D^\pm and an admissible \hat{J} arising from Floer's continuation equation.

Lemma 37. *$\mathcal{R}_{\tilde{g}}, r_{\tilde{g}}$ on cohomology do not depend on the choice of H (defining admissibility for \hat{J}). We can choose a monotone H with $\partial_s H \leq 0$ and satisfying:*

$$\begin{aligned} H : E_g &\rightarrow \mathbb{R}, \quad H|_{D^\pm \times M} = H^\pm, \\ H^- &= 0 \text{ on } D^- \times M, \\ H_t^+(y) &= g^*0 = -K_t^g(g_t(y)) \text{ on } \partial D^\pm \times M, \\ H^+ &= 0 \text{ near the centre of } D^+. \end{aligned}$$

For such H , the (j, \hat{J}) -holomorphic sections $s : S^2 \rightarrow E_g$ have $s^\pm(D^\pm) \subset M$ landing entirely in the complement of the conical end $(\Sigma \times (-\varepsilon, \infty) \times S^1, d(R\alpha))$ of (M, ω) (assuming J is conical and $K^g = \kappa R + \text{constant}$, $\kappa > 0$, on the conical end).

Proof. This is a standard cobordism argument which is proved by inspecting the 1-dimensional parts of the parametrized moduli space $\cup_\lambda \mathcal{S}(j, \hat{J}_\lambda, S)$ for a homotopy $(\hat{J}_\lambda)_{0 \leq \lambda \leq 1}$. This proves that the maps $r_{\tilde{g}}$ obtained for \hat{J}_0 and for \hat{J}_1 are chain homotopic. We omit the details.

We homotope the glued \hat{J} obtained from 5.7 to a generic \hat{J} which is admissible for a smooth monotone Hamiltonian H satisfying the claim (over D^+ we can choose an interpolation $\phi(s)K^g \circ g_t$ where ϕ is monotone: $\partial_s \phi \leq 0$, $\phi = 0$ for $s \ll 0$ (near the centre of D^+) and $\phi = -1$ near $s = 0$ (the boundary ∂D^+)).

Because H_z is monotone, the maximum principle 31 applies in the region where J is conical. So sections which touch the conical region must lie in a slice $R = \text{constant}$ (which is preserved by g_t). In this region ω is exact and so $\tilde{\Omega}$ is exact, so the holomorphic sphere $u = s : S^2 \rightarrow E_g$ would have $\int_{S^2} u^* \tilde{\Omega} = 0$. By Lemma

27, $(u^\pm)^*\tilde{\Omega} \geq 0$ pointwise, where $u^\pm : D^\pm \rightarrow M$. So $(u^\pm)^*\tilde{\Omega} = 0$. Lemma 27 also shows that u^- is constant on D^- (since $H^- = 0$ there). Via the transition, this means $t \mapsto u^+(0, t)$ along ∂D^+ is a non-constant orbit of g_t^{-1} (it is non-constant since we are assuming u does not lie in the zero section). Lemma 27 also shows $\partial_s u^+ = 0$ and hence $\partial_t u^+ = X_{H^+}$. By the first equation, the non-constant orbit $t \mapsto u^+(s, t)$ of g_t^{-1} is independent of $s \in (-\infty, 0]$. But $X_{H^+} = 0$ for $s \ll 0$, so the second equation says the orbit is constant. Contradiction. \square

6. GROMOV-WITTEN INVARIANTS

6.1. Gromov-Witten invariants. We now make some brief remarks about GW invariants, referring to [15, 22] for details.

For a closed symplectic manifold (X, ω) of dimension $\dim_{\mathbb{R}} X = 2n$, satisfying the monotonicity condition, and a generic ω -compatible almost complex structure J , the (genus 0) Gromov-Witten invariant of J -holomorphic curves $u : \mathbb{CP}^1 \rightarrow X$ with $k \geq 3$ marked points in a class $[u] = \beta \in H_2(X)$ (working over $\mathbb{Z}/2$) is

$$\text{GW}_{0,k,\beta}^X : H_*(X)^{\otimes k} \rightarrow \mathbb{Z}/2, (\alpha_1, \dots, \alpha_k) \mapsto (X_1 \times \dots \times X_k) \cdot \text{ev}_J$$

where we intersect in X^k the pseudocycle $\text{ev}_J : \mathcal{M}_{0,k}^*(\beta, J) \rightarrow X^k$ with a generic representative $X_1 \times \dots \times X_k$ of $\alpha_1 \times \dots \times \alpha_k$. Here $\mathcal{M}_{0,k}^*(\beta, J)$ is the moduli space of $PSL(2, \mathbb{C})$ -equivalence classes of stable k -pointed curves (u, z_1, \dots, z_k) , where $u : \mathbb{CP}^1 \rightarrow X$ is a simple J -holomorphic sphere in class β and z_i are pairwise distinct points in \mathbb{CP}^1 ($\phi \in PSL(2, \mathbb{C})$ acts by $(u \circ \phi^{-1}, \phi(z_1), \dots, \phi(z_k))$). To get a non-zero invariant, one requires

$$\sum \text{codim}_{\mathbb{R}}(\text{cycles}) \equiv 2mk - \sum |\alpha_i| = 2n + 2c_1(TX, \omega)(\beta) + 2k - 6.$$

To ensure ev_J is a pseudo-cycle one requires a condition on β : that β is not a multiple of a spherical homology class B with $c_1(TX, \omega)(B) = 0$ [15, Sec 6.6]. The genericity condition on $X_1 \times \dots \times X_k$ is to ensure that it is transverse to ev_J and to the evaluations maps involved in the lower strata in the compactification.

If one works over \mathbb{Q} , and one chooses differential forms $a_i \in H^{2m-|\alpha_i|}(X)$ Poincaré dual to α_i supported near X_i , then

$$\text{GW}_{0,k,\beta}^X(\alpha_1, \dots, \alpha_k) = \int_{\overline{\mathcal{M}}_{0,k}(\beta, J)} \text{ev}_1^* a_1 \wedge \dots \wedge \text{ev}_k^* a_k \in \mathbb{Q}$$

where $\overline{\mathcal{M}}_{0,k}(\beta, J)$ is the compactification by stable maps of the space of k -pointed J -holomorphic $u : \mathbb{CP}^1 \rightarrow X$ in class β , and $\sum \deg(a_i) = 2n + 2c_1(TX, \omega)(\beta) + 2k - 6$.

6.2. GW invariants counting sections of E_g . The story for (j, \hat{J}) -holomorphic sections $u : S^2 \rightarrow E_g$ is slightly different [15, Def 8.6.6]. The key observations are:

- (1) The quotient by $PSL(2, \mathbb{C})$ in the definition of the moduli spaces defining GW invariants for E_g is equivalent to imposing the condition that $u : S^2 \rightarrow E_g$ is a section, since $u \circ \phi^{-1}$ is a section for a unique $\phi = \pi_g \circ u \in PSL(2, \mathbb{C})$.
- (2) Sections lie in a class $\beta = [S^2] + j_{z_0}^* \beta_0$, for some $\beta_0 \in H_*(M)$ where $j_{z_0} : M \rightarrow E_g$ includes the fibre over z_0 . So the condition on β is automatic since $(\pi_g)_*[u] = [S^2]$, and a section is automatically simple.
- (3) Suppose we want to use *fixed* marked points $w_i \in S^2$ (pairwise distinct) and we want the sections to intersect $j_i(X_i)$ where X_i represents $\alpha_i \in H_*(M)$ and $j_i : M \rightarrow E_g$ is the inclusion of the fibre over w_i . Then, when defining

the GW invariants for E_g , we can still let the marked points $z_i \in S^2$ vary freely since the intersection condition $u(z_i) \in j_i(X_i)$ automatically forces

$$z_i = \pi_g(u(z_i)) = \pi_g(j_i(X_i)) = w_i.$$

- (4) One can make sense of these GW invariants even when $0 \leq k < 3$: we can simply add $3 - k$ extra marked points and we require the (automatically satisfied) condition that the section intersects $j_i(M)$ for these new marked points. Any section of E_g will automatically intersect $[M]$ once transversely over these new w_i . So we are ensuring the divisor axiom [15, Rmk 7.5.2].

The upshot, is that the GW invariant

$$\text{GW}_{0,k,\beta}^{E_g} : QH_*(M)^{\otimes k} \rightarrow \Lambda, (\alpha_1, \dots, \alpha_k) \mapsto (j_1(X_1) \times \dots \times j_k(X_k)) \cdot \text{ev}_J$$

corresponds precisely to the sections one plans to count modulo 2, with weight γ_β :

$$\text{GW}_{0,k,\beta}^{E_g}(\alpha_1, \dots, \alpha_k) = \#\{u \in \mathcal{S}(j, \hat{J}, \gamma_\beta + S_{\hat{g}}) : u(w_i) \in j_i(X_i)\}$$

using \hat{J} on E_g to define GW, where $\gamma_\beta \in \Gamma$ is determined by β (here $\beta = [S^2] + (j_{z_0})_*\beta_0$, and $\beta_0 \in H_2(M)$ is a spherical class so determines a $\gamma_\beta \in \Gamma$), and where we require the dimension is correct:

$$2m + 2c_1(T^v E_g, \Omega)(\gamma_\beta + S_{\hat{g}}) = \sum \text{codim}_M(X_i),$$

which is equivalent to the GW condition

$$(2m + 2) + 2c_1(T E_g, \tilde{\Omega})(\beta) + 2k - 6 = (2m + 2)k - \sum |\alpha_i|$$

where $\dim_{\mathbb{R}} M = 2m$ (using $T E_g \cong T S^2 \oplus T^v E_g$, and $6 = 2 + 2c_1(T S^2)[S^2]$).

We will only be considering the case: $k = 2$, $\alpha_1 \in QH_*^{lf}(M)$, $\alpha_2 \in QH_*(M)$.

7. NEGATIVE LINE BUNDLES

7.1. Definition and properties. Fix (B, ω_B) any closed symplectic manifold. A complex line bundle $\pi : L \rightarrow B$ is called *negative* if for some real $n > 0$,

$$c_1(L) = -n[\omega_B].$$

Examples:

- (1) $\mathcal{O}(-n) \rightarrow \mathbb{CP}^m$ for integers $n \geq 1$. Recall $\mathcal{O}(-1) = \{(x, v) : v \in x\} \subset \mathbb{CP}^m \times \mathbb{C}^{m+1}$, and $\mathcal{O}(-n) = \mathcal{O}(-1)^{\otimes n}$ has $c_1(\mathcal{O}(-n))[\mathbb{CP}^1] = -n$.
- (2) Any L dual to an ample holomorphic line bundle over a compact complex manifold B . Indeed for some $k > 0$, L^{-k} is very ample, so $L^{-k} = j^*\mathcal{O}(1)$ via the embedding $j : B \rightarrow \mathbb{CP}^m$ defined by the global holomorphic sections of L^{-k} . Let $[\omega_B] = j^*[\omega_{\mathbb{CP}^m}]$. Since the Fubini-Study form $[\omega_{\mathbb{CP}^m}] = c_1(\mathcal{O}(1))$,

$$-kc_1(L) = c_1(L^{-k}) = j^*c_1(\mathcal{O}(1)) = j^*[\omega_{\mathbb{CP}^m}] = [\omega_B].$$

Indeed any compact complex manifold admitting a holomorphic embedding $B \subset \mathbb{CP}^m$ arises in this way, and by Kodaira's embedding theorem these are precisely the compact Kähler manifolds with integral Kähler form.

Lemma 38 (see Oancea [16]). *$L \rightarrow B$ is negative iff L admits a Hermitian metric, and some Hermitian connection whose curvature \mathcal{F} satisfies $\frac{i}{2\pi}\mathcal{F}(v, J_B v) < 0$ for all $v \neq 0 \in TB$ and for all almost complex structures J_B compatible with ω_B (meaning $\omega_B(\cdot, J_B \cdot)$ is a metric).*

This Lemma essentially follows from the fact that $\frac{i}{2\pi}\mathcal{F}$ represents $c_1(L)$ inside $H^2(B; \mathbb{R})$. For example, in one direction, if $c_1(L) = -n[\omega_B]$, then there is a Hermitian metric on L whose curvature satisfies $n\omega_B = \frac{1}{2\pi i}(\mathcal{F} + da)$, and by adding the one-form $-a$ to the connection one can get rid of the exact term da .

7.2. Construction of the symplectic form. From now on M is the total space of a negative line bundle $\pi : L \rightarrow (B, \omega_B)$, and we assume a connection and metric as above are chosen. Thus

$$c_1(L) = [\frac{i}{2\pi}\mathcal{F}] = -n[\omega_B] \in H^2(B, \mathbb{Z}) \cap H^2(B, \mathbb{R}).$$

We choose $\Sigma = \{r = 1\}$ to be the hypersurface for M (which will be contact).

Examples. $\mathcal{O}(-1) \rightarrow \mathbb{C}P^m$ arises as the blow-up of \mathbb{C}^{m+1} at the origin, so $\Sigma \cong S^{2m+1}$ is the preimage of $S^{2m+1} \subset \mathbb{C}^{m+1}$. The multiplication action on \mathbb{C}^{m+1} by a primitive n -th root of unity lifts to the blow-up, fixing the exceptional $\mathbb{C}P^m$ which is the zero section of $\mathcal{O}(-1)$. The quotient by this action defines a bundle map $\mathcal{O}(-1) \rightarrow \mathcal{O}(-n)$. So for $\mathcal{O}(-n) \rightarrow \mathbb{C}P^m$, $\Sigma = S^{2m+1}/(\mathbb{Z}/n)$ is a Lens space.

We will now construct the symplectic form ω for M of the form

$$\omega = d\theta + \varepsilon\Omega \quad (\text{fixed } \varepsilon > 0)$$

consisting of a *non-exact* form $[d\theta] = n\pi^*[\omega_B]$ (only away from the zero section it is exact) and a term Ω which is fibrewise the area form (not contributing to $[\omega]$).

For $w \in L$, define the radial function r by $r(w) = |w|$ in the above metric.

The connection defines the fibrewise angular 1-form $\theta = \frac{1}{4\pi}\pi^*d^c \log r^2$ on $L \setminus$ (zero section), which satisfies

$$d\theta = -\frac{1}{2\pi i}\pi^*\partial\bar{\partial} \log r^2 \equiv -\pi^*c_1^{\mathbb{C}}(L) = -\frac{i}{2\pi}\pi^*(\mathcal{F}) = n\pi^*\omega_B.$$

Explicitly [1, p.132], $\theta_w(\cdot) = \frac{1}{2\pi r^2}\langle iw, \cdot \rangle$ so in the complement of the zero section

$$\theta_w(w) = 0, \quad \theta_w(iw) = 1/2\pi$$

where w, iw is considered as a basis of $T_w L \cong L_w$, and $\theta = 0$ on horizontal vectors.

Lemma 39. $d\theta(v, \cdot) = 0$ for any vertical vector $v \in TL$ ($v \in \ker d\pi$). On horizontal $v, v' \in T_w L$, $d\theta(v, v') = -\theta([v, v']) = \theta_w(\mathcal{F}_{d\pi \cdot v, d\pi \cdot v'} w)$ (see [1, p.133]). Since $\pi^*\mathcal{F}$ is imaginary valued, we deduce $d\theta = \frac{1}{2\pi i}\pi^*\mathcal{F}$, which extends $d\theta$ over the zero section.

Remark: our curvature is opposite to [1, p.120].

On $L \setminus$ (zero section) define $\Omega = d(r^2\theta)$. Fibrewise this is (area form)/ π , so extend Ω over the zero section by

$$\Omega|_{\text{fibre}} = (\text{area form})/\pi \quad \Omega(T(\text{zero section}), \cdot) = 0.$$

7.3. Liouville and Reeb fields.

Lemma 40. Fibrewise the Liouville and Reeb fields for Ω at $w \in L$ are

$$Z_\Omega = \frac{1}{2}w, \quad Y_\Omega = 2\pi iw = 4\pi i Z_\Omega.$$

Proof. By Lemma 39, $d(r^2\theta)(\frac{w}{2}, \cdot) = 2r dr(\frac{w}{2})\theta = (2r^2/2)\theta = r^2\theta$ using $dr(w) = r$ and $\theta(w) = 0$; $r^2\theta(2\pi iw) = 1$ on Σ , $d(r^2\theta)(2\pi iw, \cdot) = 0$ on $T\Sigma$ using $dr(iw) = 0$ and $dr(T\Sigma) = 0$ (by Lemma 39, $d\theta(iw, \cdot) = 0$ since iw is vertical). \square

Now study the conical symplectic manifold (M, ω) with hypersurface Σ , where

$$\omega = d\theta + \varepsilon\Omega = d((1 + \varepsilon r^2)\theta) \quad (\text{fix } \varepsilon > 0).$$

At infinity, indeed in the complement of the zero section, ω is exact since the primitive $(1 + \varepsilon r^2)\theta$ is defined there.

Lemma 41. *The Liouville field Z for (M, ω) is*

$$Z = \frac{1 + \varepsilon r^2}{\varepsilon r^2} \cdot \frac{w}{2}$$

which is defined away from the zero section and is outward pointing along Σ .

The Reeb vector field is

$$Y = \frac{2\pi}{1 + \varepsilon} iw.$$

The Reeb periods are $k(1 + \varepsilon)$ for $k = 0, 1, 2, \dots$, with a Reeb orbit $w(t) = e^{2\pi it/(1 + \varepsilon)} w_0$ in each fibre with base point w_0 and $t \in [0, k(1 + \varepsilon)]$.

Proof. By the previous two Lemmas, $\omega(Z_\Omega, \cdot) = \varepsilon r^2 \theta$. So normalizing: $Z = \frac{1 + \varepsilon r^2}{\varepsilon r^2} Z_\Omega$. Since Y_Ω is vertical, by Lemma 39 we have $d\theta(Y_\Omega, \cdot) = 0$ and $d\Omega(Y_\Omega, \cdot) = -2rdr(\cdot)$ (using $\theta(iw) = 1/2\pi$). So $\omega(Y_\Omega, \cdot) = 0$ on $T\Sigma$ (parallel transport preserves r , so $T\Sigma_w$ is spanned by the horizontal vectors and the vertical iw , and $dr(iw) = 0$). Finally, on Σ , $(\theta + \varepsilon r^2 \theta)(Y_\Omega) = 1 + \varepsilon$. So normalizing: $Y = Y_\Omega/(1 + \varepsilon)$. \square

7.4. Conical parametrization.

Lemma 42. *The radial coordinate R in the sense of Section 2.1 is*

$$R = \frac{1 + \varepsilon r^2}{1 + \varepsilon},$$

defined on all of M with differential $dR = (2\varepsilon r)(1 + \varepsilon)^{-1} dr$ vanishing on the zero section. The flow of Z defines the conical parametrization

$$(M_1, \omega|_{M_1}) \cong \left(\Sigma \times \left(\frac{1}{1 + \varepsilon}, \infty \right), d(R\alpha) \right)$$

where R is the coordinate for the interval, $\alpha = (1 + \varepsilon)\theta|_\Sigma$, $M_1 = M \setminus (\text{zero section})$.

Proof. Let $w(t)$ solve $\dot{w}(t) = Z(w(t))$ with $w(0) = w_0 \in \Sigma$. The radial coordinate is defined by $R(w(t)) = e^t$. The solution w is unique, and we try to solve for $w(t) = r(t)w_0$. Then the equation becomes $\dot{r} = (1 + \varepsilon r^2)/2\varepsilon r$. So $\partial_t(1 + \varepsilon r^2) = 2\varepsilon r\dot{r} = 1 + \varepsilon r^2$, thus $1 + \varepsilon r^2 = (1 + \varepsilon)e^t = (1 + \varepsilon)R$. \square

7.5. The Hamiltonians.

$$H = h_k(R) = k(1 + \varepsilon)R.$$

Since in general $X_H = h'(R)Y$, we obtain

$$X_H = k(1 + \varepsilon)Y.$$

The flow is $w(t) = e^{k2\pi it} w(0)$. Observe that for integer values of k the flow is 1-periodic, but for non-integer values of k the only orbits are the constant orbits lying on the zero section (which is the critical level set for H).

The Hamiltonians h_k , $k \notin \mathbb{Z}$, have degenerate 1-orbits, indeed they are Morse-Bott with critical level set C the zero section.

There are two ways around this. One can introduce an auxiliary Morse function f on C , and then one defines $CF^*(h_k, f)$ by standard Morse-Bott techniques (see

for example Bourgeois-Oancea [4]). The generators will be the critical points of f in C , and the differential will count rigid trajectories which are suitable combinations of $-\nabla f$ -flowlines inside C and Floer flowlines with ends on C . This approach is an infinitesimal version of the second approach, which is to explicitly construct a perturbation of the form

$$h_{k,\epsilon} = h_k + \epsilon f$$

using a time-dependent function f supported near C and Morse on C , and a small enough constant $\epsilon > 0$. For small enough ϵ , one then shows that the local Floer cohomology near C is isomorphic to the Morse cohomology of C . This is also a standard method (for instance, for S^1 critical level sets, see [8, Prop. 2.2]). We omit these details.

7.6. The g -action. The action by rotation in the fibres,

$$g_t = e^{2\pi i t},$$

is Hamiltonian generated by $K = h_1(R) = (1 + \epsilon)R$. Since g_t preserves R , the pull-back of the Hamiltonians by the g -action is:

$$g^* h_k = h_k \circ g_t - K \circ g_t = (1 + \epsilon)kR - (1 + \epsilon)R = h_{k-1}.$$

7.7. Complex structure. The complex structure $J = i$ does not strictly satisfy “ $JZ = Y$ ”, but it satisfies a rescaled version:

$$Y = \frac{4\pi\epsilon r^2}{(1 + \epsilon)(1 + \epsilon r^2)} JZ \Big|_{\Sigma} = \frac{4\pi\epsilon}{(1 + \epsilon)^2} JZ,$$

so the contact condition “ $dR \circ J = -R\alpha$ ” is actually rescaled as follows:

$$dR \circ J = \frac{-4\pi\epsilon r^2}{(1 + \epsilon)(1 + \epsilon r^2)} R\alpha.$$

Lemma 43. *The maximum principle 31 holds for $J = i$ everywhere on M .*

Proof. We mimick the old proof (Lemma 31). Let $\rho = (1 + \epsilon r^2) \circ u$. Since $dr|_w(w) = r, dr|_w(iw) = 0$, we deduce

$$dr \circ i = -2\pi r \theta.$$

Thus, letting $\tilde{\theta} = (1 + \epsilon r^2)\theta$ denote the primitive for ω ,

$$\begin{aligned} d\rho \circ j &= 2\epsilon r dr(i \circ du + \nu \circ j) \\ &= \frac{4\pi\epsilon r^2}{1 + \epsilon r^2} (-u^* \tilde{\theta} + dt \otimes \tilde{\theta}(X_H)) \\ &= 4\pi \frac{\rho - 1}{\rho} (-u^* \tilde{\theta} + dt \otimes \tilde{\theta}(X_H)) \end{aligned}$$

$$(-d(d\rho \circ j) + 1^{st} \text{ order in } \rho) \geq 4\pi \frac{\rho - 1}{\rho} (-(R \circ u) \partial_s h') ds \wedge dt - d(4\pi \frac{\rho - 1}{\rho}) \wedge \frac{\rho}{4\pi(\rho - 1)} (d\rho \circ j).$$

We need to ensure the right hand side is a positive multiple of $ds \wedge dt$ so that, as in the old proof, $(\Delta\rho + 1^{st} \text{ order terms in } \rho) \geq 0$ provided $\partial_s h' \leq 0$.

So we need $\rho \geq 1$ for the first term. The second term is $-\frac{1}{\rho^2} \cdot \frac{\rho}{\rho - 1} d\rho \wedge d\rho \circ j$, and $d\rho \wedge d\rho \circ j = -(\partial_s \rho)^2 - (\partial_t \rho)^2 ds \wedge dt$. So $\rho \geq 1$ suffices, equivalently: $r \geq 0$. \square

Corollary 44. *If H is monotone as in 5.8, and \hat{J} is admissible with $J = i$, then (j, \hat{J}) -holomorphic sections of $E_g \rightarrow S^2$ must land in the zero sections of the fibres.*

Proof. Lemmas 37 and 43, using that ω is exact except on the zero section. \square

Lemma. *The (non-admissible) complex structure $\hat{J} = \begin{bmatrix} j & 0 \\ 0 & i \end{bmatrix}$ on $D^\pm \times M$ yields a complex structure on E_g (i is g -invariant) and it can be used to compute $r_{\tilde{g}}, \mathcal{R}_{\tilde{g}}$ possibly after a generic small perturbation to make it regular.*

Proof. Let $\hat{J}_H = \begin{bmatrix} j & 0 \\ ds \otimes X_H - dt \otimes J_z X_H & i \end{bmatrix}$ constructed for the monotone H as in 5.8. If H is the same as the Hamiltonian defining $\tilde{\Omega}$, then we showed in Lemma 26 that \hat{J}_H is compatible with a symplectic form $\tilde{\Omega} + \pi_g^* \sigma$.

For \hat{J}_0 (the \hat{J} of the claim), compatibility will fail at infinity but it will still hold in a large compact region surrounding the zero section of E_g (which can be made larger by rescaling σ by a positive constant).

However, for the purpose of defining $\mathcal{R}_{\tilde{g}}, r_{\tilde{g}}$, this lack of compatibility will not matter if we can show that all (j, \hat{J}_{H_λ}) -holomorphic sections lie in a compact region where compatibility holds, for each H_λ in a homotopy $(H_\lambda)_{0 \leq \lambda \leq 1}$ from H to 0.

Inspecting the proof of Lemma 31 or 43, the new term in $d\rho \circ j$ caused by changing \hat{J} (but keeping $\tilde{\Omega}$ the same) is the term $dt \otimes \theta(X_{H_\lambda - H})$. So, assuming $H_\lambda - H$ is radial, say $(H_\lambda - H)(u) = k_\lambda(\rho)$, the new term in $-d(d\rho \circ j)$ is

$$-d(\rho k_\lambda(\rho))dt = -(k_\lambda(\rho) - \rho k'_\lambda(\rho)) \partial_s \rho ds \wedge dt$$

and these first order terms in ρ don't affect the proof of the maximum principle.

By 5.8, $\mathcal{R}_{\tilde{g}}, r_{\tilde{g}}$ will not be affected in homology if we homotope \hat{J}_H to \hat{J}_0 . \square

Remark. *In the notation of the proof, if \hat{J}_0 is not regular then one needs to homotope it to \hat{J}_L , where L is a small perturbation of 0 typically non-radial near the zero section (the maximum principle will not hold there, so (j, \hat{J}_L) -holomorphic sections may not lie entirely in the zero section) but $L = 0$ away from the zero section (so the maximum principle applies and sections cannot touch this region).*

7.8. The choice of \tilde{g} . The action of g on $\mathcal{L}_0 M$ lifts to an action of $\widetilde{\mathcal{L}_0 M}$. We choose the lift \tilde{g} so that the constant orbits x on the zero section lifted to $(c_x, x) \in \widetilde{\mathcal{L}_0 M}$ satisfy $\tilde{g} \cdot (c_x, x) = (c_x, x)$, where $c_x : D \rightarrow M$ is the constant map to x . So $S_{\tilde{g}}$ is represented by the constant $s_g^+(z) = c_x(z) = x$, $s_g^-(z) = (\tilde{g}c_x)(z) = c_x(z) = x$.

Lemma. $\tilde{\Omega}(s_{\tilde{g}}) = 0$.

Proof. As in 5.8, choose $\tilde{\Omega}^+ = \omega^+ + \phi'(s)K \circ g_t ds \wedge dt$, $\tilde{\Omega}^- = \omega^-$ so $\tilde{\Omega}(s_{\tilde{g}}) = \int_{D^+} \phi'(s)K(x) ds \wedge dt = 0$ since $K = 0$ on the zero section. \square

Lemma 45. $I(\tilde{g}) = 1$ (defined in Section 3.1).

Proof. Using any (c_x, x) as above, pick a unitary trivialization of L over a neighbourhood of the point $b = \pi(x) \in B$ to obtain

$$\tau_{c_x} : x^* TM \cong S^1 \times T_x M \cong S^1 \times T_b B \times L_b \cong S^1 \times T_b B \times \mathbb{C}.$$

Now $\tilde{g} \cdot (c_x, x) = (c_x, x)$, and g_t is a linear holomorphic action given by multiplication by a complex number, so dg_t commutes with $\tau_{c_x}(t)$. Thus

$$\ell(t) = \tau_{c_x}(t) \circ dg_t \circ \tau_{c_x}(t)^{-1} = dg_t \circ \tau_{c_x}(t) \circ \tau_{c_x}(t)^{-1} = e^{2\pi i t},$$

so $\ell(t)$ is the rotation of the \mathbb{C} factor and the identity on the $T_b B$ factor. So $t \mapsto \det e^{2\pi i t} = e^{2\pi i t}$ is 1 in $H_1(S^1; \mathbb{Z}) \cong \mathbb{Z}$. So $I(\tilde{g}) = \deg(\ell) = 1$. \square

8. SYMPLECTIC COHOMOLOGY OF $M = \text{Tot}(\mathcal{O}(-n) \rightarrow \mathbb{CP}^1)$

8.1. **The $r_{\tilde{g}}$ map for $\mathcal{O}(-n)$.** Consider $M = \text{Tot}(\mathcal{O}(-n) \rightarrow \mathbb{CP}^1)$ for $n \geq 1$. The generators of $H_*^{lf}(M)$ are in degree 2 and 4:

$$\begin{aligned} F &= \text{fibre } \mathbb{C}, \text{ Poincaré dual to the zero section } [\mathbb{CP}^1] \\ M &= \text{fundamental chain, Poincaré dual to the point class } [\text{pt}] \end{aligned}$$

Using a connection, $TM \cong T\mathbb{CP}^1 \oplus \mathcal{O}(-1)$, so $c_1(TM)[\mathbb{CP}^1] = 2 - n$. The zero section $[\mathbb{CP}^1]$ generates $\pi_2(M)$. M satisfies weak⁺ monotonicity: it is either monotone (for $\mathcal{O}(-1)$), or $c_1 = 0$ (for $\mathcal{O}(-2)$), or the min Chern number $|N| \geq 1$:

$$N = c_1(TM, \omega)([\mathbb{CP}^1]) = 2 - n.$$

Moreover, Λ is generated by $[\mathbb{CP}^1]$ which has $c_1(TM, \omega)[\mathbb{CP}^1] = N$, $\omega[\mathbb{CP}^1] > 0$. Writing $t = [\mathbb{CP}^1]$ for the generator of Λ , and $t^m = m[\mathbb{CP}^1]$, we obtain

$$\begin{aligned} \Lambda &= \mathbb{Z}[t^{-1}, t] = \left\{ \sum n_j t^{m_j} : n_j \in \mathbb{Z}/2, \lim_{j \rightarrow \infty} m_j = \infty \right\} \\ |t| &= -2c_1(TM, \omega)[\mathbb{CP}^1] = -2N \quad (\text{homological grading}) \end{aligned}$$

By Lemma 45 and the choice of \tilde{g} in 7.8, $c_1(TE_{\tilde{g}}^v, \Omega)(S_{\tilde{g}}) = -I(\tilde{g}) = -1$, and $\tilde{\Omega}(S_{\tilde{g}}) = 0$. So the dimension of the space of sections (Lemma 32) is

$$\begin{aligned} \dim \mathcal{S}(j, \hat{J}, t^m + S_{\tilde{g}}) &= 2 \dim_{\mathbb{C}} M + 2c_1(TE_{\tilde{g}}^v, \Omega)(S_{\tilde{g}}) + 2m c_1(TM, \omega)(t) \\ &= 2 + 2N \cdot m \end{aligned}$$

The condition that the sections intersect F or M at z_0 cuts down the dimension respectively by 2 or 0, and then evaluation at z_{∞} sweeps out a locally finite chain in dimension $2Nm$ or $2 + 2Nm$. So in these two cases, the possibilities are:

$-n$	$N = 2 - n$	$ t = -2N$	$2Nm$	$2 + 2Nm$	$l.f. \ 2Nm\text{-chains}$	$l.f. \ (2 + 2Nm)\text{-chains}$
-1	1	-2	$2m$	$2 + 2m$	$\mathbf{F}(m=1), \mathbf{M}(m=2)$	$\mathbf{F}(m=0), \mathbf{M}(m=1)$
-2	0	0	0	2	none	$\mathbf{F}(\text{any } m)$
-3	-1	2	$-2m$	$2 - 2m$	$\mathbf{F}(m=-1), \mathbf{M}(m=-2)$	$\mathbf{F}(m=0), \mathbf{M}(m=-1)$
-4	-2	4	$-4m$	$2 - 4m$	$\mathbf{M}(m=-1)$	$\mathbf{F}(m=0)$
≤ -5	≤ -3	≥ 6	$\leq -6m$	$\leq 2 - 6m$	none	$\mathbf{F}(m=0)$

We can rule out $m < 0$ since a (j, \hat{J}) -holomorphic section S has $\tilde{\Omega}(S) \geq 0$ (by Lemma 27) and $\tilde{\Omega}(t^m + S_{\tilde{g}}) = m\omega(\mathbb{CP}^1) + \tilde{\Omega}(S_{\tilde{g}}) = m\omega(\mathbb{CP}^1)$. The sections s for $m = 0$ are constant (since $\tilde{\Omega}(s) = 0$).

The sections in class $t^m + S_{\tilde{g}}$ contribute with Novikov weight t^m to $r_{\tilde{g}}$. Thus viewing $\Lambda^2 \equiv QC_*^{lf}(M) \equiv \Lambda \cdot (F \otimes 1) + \Lambda \cdot (M \otimes 1)$, the matrix $r_{\tilde{g}} : \Lambda^2 \rightarrow \Lambda^2$ is

$$\begin{array}{c|c|c} n=1 & n=2 & n \geq 3 \\ \hline \begin{bmatrix} At & C \\ Bt^2 & Dt \end{bmatrix} & \begin{bmatrix} 0 & C\lambda \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix} \end{array}$$

where $A, B, C, D \in \mathbb{Z}/2$, $\lambda \in \Lambda$. Note this is nilpotent for $n \geq 2$, so:

Corollary. $SH^*(M) = 0$ for $M = \text{Tot}(\mathcal{O}(-n) \rightarrow \mathbb{CP}^1)$ and $n \geq 2$.

8.2. Description of $\mathbf{E}_{\mathbf{g}}$ for $\mathbf{M} = \text{Tot}(\mathcal{O}(-1) \rightarrow \mathbb{CP}^1)$.

Lemma. The \mathbb{C} -line bundle over \mathbb{CP}^1 with transition $\partial D^+ \times \mathbb{C} \rightarrow \partial D^- \times \mathbb{C}$ given by $([e^{2\pi it} : 1], x) \rightarrow ([1 : e^{-2\pi it}], g_t \cdot x)$ is the bundle $\mathcal{O}(-1)$ (where $g_t = e^{2\pi it}$).

Proof. Coordinates: $[w : 1]$ on $D^+ =$ Northern hemisphere of $S^2 \equiv \mathbb{CP}^1$, and $[1 : z]$ on $D^- =$ Southern hemisphere. Claim: $\mathcal{O}(-1)$ is defined by the transition $([w : 1], x) \mapsto ([1 : \frac{1}{w}], wx)$. Sanity check: $\mathcal{O}(1)$ has transition $g^{-1} = 1/w$ and has a holomorphic section $w = 1$ on D^+ , $z = z$ on D^- (simple zero at z_0).

We compute c_1 . The orientation on D^+ is induced by $(s, t) \in (-\infty, 0] \times \mathbb{R}$ via $w = e^{2\pi(s+it)}$. The equator $C = \{[e^{2\pi it} : 1]\}$ is the positively oriented boundary of D^+ : (outward normal, ∂_t) is an oriented basis of S^2 . The equator is a *negatively* oriented boundary for D^- , so $c_1[\mathbb{CP}^1]$ is $-\deg(\text{transition from } \partial D^+ \text{ to } \partial D^-)$, and $-\deg(t \mapsto e^{2\pi it} \in U(1)) = -1$. \square

Corollary. *For $M = \text{Tot}(\pi_M : \mathcal{O}(-1) \rightarrow \mathbb{CP}^1)$, the complex line bundle $(\pi_g, \pi_M) : E_g \rightarrow S^2 \times \mathbb{CP}^1$ is $\mathcal{O}(-1, -1) = \pi_g^* \mathcal{O}(-1) \otimes \pi_M^* \mathcal{O}(-1)$.*

Proof. The transition along the equator of S^2 is as in the previous lemma, and the transition over the equator of \mathbb{CP}^1 is the same as the transition for $M = \mathcal{O}(-1)$. \square

Lemma. *$m = d = \text{degree}(\text{sections in class } t^m + S_{\bar{g}})$ so the virtual dimension of the space of sections in class $(1, d) \in H^2(S^2 \times \mathbb{CP}^1)$ via (π_g, π_M) is $2 + 2d$.*

Proof. Viewing $E_g = \text{Tot}(\mathcal{O}(-1, -1) \rightarrow S^2 \times \mathbb{CP}^1)$, a choice of connection yields $TE_g \cong T(S^2 \times \mathbb{CP}^1) \oplus \mathcal{O}(-1, -1)$, so

$$c_1(TE_g) = (2, 2) + (-1, -1) = (1, 1) \in H^2(S^2 \times \mathbb{CP}^1)$$

Similarly, using $E_g \rightarrow S^2$, $TE_g \cong TS^2 \oplus T^v E_g$ so

$$c_1(T^v E_g) = c_1(TE_g) - c_1(TS^2) = (1, 1) - (2, 0) = (-1, 1) \in H^2(S^2 \times \mathbb{CP}^1).$$

The space of sections in class $(1, d)$ therefore has $\dim = 4 + 2 \cdot \langle (-1, 1), (1, d) \rangle = 4 - 2 + 2d$. Compare this with the formula $4 - 2 + 2m$ for sections in class $t^m + S_{\bar{g}}$. \square

Remark. *For $M = \text{Tot}(\mathcal{O}(-n) \rightarrow \mathbb{CP}^1)$, $(E_g \rightarrow S^2 \times \mathbb{CP}^1) = \mathcal{O}(-1, -n)$ and $m = d$.*

8.3. The sections of E_g for $M = \mathcal{O}(-1) \rightarrow \mathbb{CP}^1$. In

$$r_{\bar{g}} = \begin{bmatrix} At & C \\ Bt^2 & Dt \end{bmatrix}$$

only $m = 0, 1, 2$ contribute, so we only care about sections in classes $(1, 0), (1, 1), (1, 2)$.

Sections in class $(1, 0)$ have area $\tilde{\Omega}(S_{\bar{g}}) = 0$, so they are constant sections:

$$u : S^2 \rightarrow S^2 \times \mathbb{CP}^1 \subset E_g, z \mapsto (z, y),$$

some $y \in \mathbb{CP}^1$. This is a 2-dimensional space of sections, agreeing with $\text{vir} \dim_{\mathbb{R}} = 2$.

Lemma 46. *$\hat{J} = \begin{bmatrix} j & 0 \\ 0 & i \end{bmatrix}$ is regular for the constant sections, and $C = -1$.*

Proof. We are in the integrable case, so D_u is just the Dolbeaut operator:

$$\bar{\partial} = \partial_s + J\partial_t : \Gamma(u^* T^v E_g) \rightarrow \Gamma(u^* T^v E_g \otimes_{\mathbb{C}} \Omega^{0,1} S^2), D_u \cdot \xi = (\partial_s u + J\partial_t u) \otimes (ds - i dt),$$

(we only differentiate in the vertical directions of E_g since we only consider sections).

Now $(u^* T^v E_g)_z = T(E_g)_{(z, y)} \cong T_y \mathbb{CP}^1 \oplus (\mathcal{O}(-1))_y \cong \mathbb{C} \oplus \mathbb{C}$. The transition over the equator of S^2 is multiplication by dg_t , which acts by (id, g_t) on the fibre $\mathbb{C} \oplus \mathbb{C}$. Thus, as bundles over S^2 , $u^* T^v E_g \cong \underline{\mathbb{C}} \oplus \mathcal{O}(-1)$. We deduce:

$$\bar{\partial} : \Gamma(S^2, \underline{\mathbb{C}} \oplus \mathcal{O}(-1)) \rightarrow \Omega^{0,1}(S^2, \underline{\mathbb{C}} \oplus \mathcal{O}(-1))$$

Using Dolbeaut's theorem $H_{\bar{\partial}}^{p,q}(S^2, \underline{\mathbb{C}} \oplus \mathcal{O}(-1)) \cong H^q(S^2, \Omega^p(\underline{\mathbb{C}} \oplus \mathcal{O}(-1)))$, and using Serre duality (for the canonical bundle $T^* S^2 = \mathcal{O}(-2)$),

$$\begin{aligned} \text{coker } \bar{\partial} &= H_{\bar{\partial}}^{0,1}(S^2, \underline{\mathbb{C}} \oplus \mathcal{O}(-1)) \cong H^1(S^2, \mathcal{O}(\underline{\mathbb{C}} \oplus \mathcal{O}(-1))) \\ &\cong H^0(S^2, (\mathcal{O}(\underline{\mathbb{C}} \oplus \mathcal{O}(-1)))^\vee \otimes T^* S^2)^\vee \\ &\cong H^0(S^2, \mathcal{O}(-2))^\vee \oplus H^0(S^2, \mathcal{O}(-1))^\vee = 0. \end{aligned}$$

since $\mathcal{O}(-k)$ has no global holomorphic sections for $k \geq 1$. So D_u is surjective, so \hat{J} is regular for the constants. A small perturbation of \hat{J} to make the other moduli spaces regular will not affect the count of constants, so to find C we can use \hat{J} .

C is the multiple of $[F] \in H_*^{lf}(M)$ corresponding to the chain swept out by evaluation at z_∞ of the constant sections intersecting $[M]$ at z_0 . The latter condition is void, so the chain is $[\mathbb{C}P^1] \in H_*^{lf}(M)$. The intersection pairing $H_*(M) \otimes H_{4-*}^{lf}(M) \rightarrow \mathbb{Z}$ maps $[\mathbb{C}P^1] \otimes [F] \mapsto 1$ and $[\mathbb{C}P^1] \otimes [\mathbb{C}P^1] \mapsto [\mathbb{C}P^1] \bullet [\mathbb{C}P^1] = -1$. So $[\mathbb{C}P^1] = -[F] \in H_2^{lf}(M)$. Thus $C = -1$. \square

Remark. For $\mathcal{O}(-n)$, regularity is proved in the same way, so $C = -n = c_1(\mathcal{O}(-n))$.

Lemma 47. *Sections in class $(1, d)$ for $d \geq 1$ form a moduli space isomorphic to $\mathcal{M}(\mathbb{P}^1 \times \mathbb{P}^1; \beta = (1, d))$: the rational curves in $\mathbb{P}^1 \times \mathbb{P}^1$ in class $(1, d)$ (abbreviating $\mathbb{P}^1 = \mathbb{C}P^1$) quotiented by $PSL(2, \mathbb{C})$ reparametrization. Let $Z = \mathbb{P}^1 \times \mathbb{P}^1$. We expect an obstruction bundle of $\text{rank}_{\mathbb{R}} = 2d$ since:*

$$\begin{aligned} \dim \mathcal{M}(Z; (1, d)) &= 2(\dim_{\mathbb{C}} Z + c_1(Z)(\beta) - 3) = 2(2 + 2 + 2d - 3) = 2 + 4d \\ \text{vir} \dim \mathcal{M}(E_g; (1, d)) &= 2 + 2d. \end{aligned}$$

Proof. Sections in class $(1, 1)$ yield a degree 1 holomorphic map $\pi_M \circ s : S^2 \rightarrow \mathbb{C}P^1$, because $\pi_M : M \rightarrow \mathbb{C}P^1$ is (\hat{J}, j) holomorphic since $\nu \circ j$ lands in the vertical tangent space of M . We quotient by the $PSL(2, \mathbb{C})$ reparametrizations $u \mapsto u \circ \phi^{-1}$ to ensure \mathbb{P}^1 maps identically onto the first factor. \square

Lemma 48. $r_{\bar{g}} = \begin{bmatrix} A & -1 \\ 0 & 0 \end{bmatrix}$, where A is the count of holomorphic sections $S^2 \rightarrow E_g$ in the class $(1, 1)$ (after perturbing J to achieve regularity) which intersect F over z_0 and a (perturbed) $\mathbb{C}P^1$ over z_∞ .

Proof. The entries B, D involve a count of sections which have some intersection condition at z_0 and which sweep out a multiple of $[M]$ under evaluation at z_∞ . However, even after perturbing J to achieve regularity of the moduli space of sections, the maximum principle implies that the sections all land in a certain compact subset of E_g . So evaluation at z_∞ involves a bounded lf chain in M . The multiple of $[M]$ is determined via Poincaré duality by intersecting with the point class. In homology, it does not matter which point we choose, so we can pick a point outside that compact subset of M , thus avoiding the bounded lf chain. So $B = D = 0$.

The entry A involves the intersection condition F at z_0 , and $\mathbb{C}P^1$ at z_∞ ($\mathbb{C}P^1$ is the cycle dual to the lf cycle F via intersection product). \square

8.4. Calculation of A using obstruction bundles. In our setup, for $M = \text{Tot}(\mathcal{O}(-1) \rightarrow \mathbb{C}P^1)$, we want to count sections in class $\beta = (1, 1)$:

$$\begin{aligned} A &= \text{GW}_{0,2,\beta=(1,1)}^{E_g}((j_{z_0})_*[F], (j_{z_\infty})_*[\mathbb{C}P^1]) \\ &= \#\{u \in \mathcal{S}(j, \hat{J}, t + S_{\bar{g}}) : u(z_0) \in j_{z_0}(F), u(z_\infty) \in j_{z_\infty}(\mathbb{C}P^1)\} \end{aligned}$$

The standard J on the fibre M yields a non-regular $\hat{J} = \begin{bmatrix} j & 0 \\ 0 & J \end{bmatrix}$ for the moduli space of sections in class $(1, 1)$ by Lemma 47, with $\text{rank}_{\mathbb{R}} = 2$ obstruction bundle

$$(\text{Obs} = \text{coker } D_u) \rightarrow \mathcal{M}_{\hat{J}}$$

$$\mathcal{M}_{\hat{J}} = \{u \in \mathcal{M}(1, 1) \cong PSL(2, \mathbb{C}) : u(z_0) \in j_{z_0}(F), u(z_\infty) \in j_{z_\infty}(P)\}$$

where D_u is the linearization of the $\bar{\partial}_{\hat{J}}$ operator defining (j, \hat{J}) -holomorphic sections, and where F is a generic fibre of M and P is a perturbation of \mathbb{CP}^1 (perturbing smoothly in the vertical direction, it will intersect the zero section of M in a point).

Lemma 49. *Assuming that we can extend the obstruction bundle smoothly over a smooth compactification of $\overline{\mathcal{M}}_{\hat{J}}$ (for which the tangent spaces are the kernels $\ker D_u$), then the coefficient A in Lemma 48 is*

$$A = \text{GW}_{0,2,(1,1)}^{E_g}(j_{z_0}F_1, j_{z_\infty}\mathbb{CP}^1) = \langle e(\overline{\text{Obs}}), \overline{\mathcal{M}}_{\hat{J}} \rangle.$$

Proof. We already discussed the first equality. The second equality is a standard cobordism argument analogous to [15, Sec 7.2]. The idea is that one constructs a smooth family of bundles $\overline{\text{Obs}}_{\hat{J}_t} \rightarrow \overline{\mathcal{M}}_{\hat{J}_t}$ such that $\bar{\partial}_{\hat{J}_t}$ lands in $\overline{\text{Obs}}_{\hat{J}_t}$, starting at the given bundle at $t = 0$ with $\hat{J}_0 = \hat{J}$, and ending at $t = 1$ with a regular admissible \hat{J}_1 . By construction, the zero set of $\bar{\partial}_{\hat{J}_1}$ is the count of (j, \hat{J}_1) -holomorphic sections of E_g in class $(1, 1)$ intersecting F, P over z_0, z_∞ , since \hat{J}_1 is regular. The Euler number $\langle e(\overline{\text{Obs}}_{\hat{J}_t}), \overline{\mathcal{M}}_{\hat{J}_t} \rangle$ is constant in t , and at $t = 1$ equals the count of zeros of a section (such as $\bar{\partial}_{\hat{J}_1}$) transverse to the zero section. Hence

$$A = \langle e(\overline{\text{Obs}}_{\hat{J}_1}), \overline{\mathcal{M}}_{\hat{J}_1} \rangle = \langle e(\overline{\text{Obs}}), \overline{\mathcal{M}}_{\hat{J}} \rangle.$$

The family is constructed by choosing a homotopy from $\hat{J}_0 = \hat{J}$ to a regular admissible \hat{J}_1 in a neighbourhood of \hat{J} inside the space \mathcal{J} of admissible almost complex structures on E_g . The family lives over \hat{J}_t inside the larger bundle obtained by extending $\overline{\text{Obs}} \rightarrow \overline{\mathcal{M}}$ over a product neighbourhood W of $\overline{\mathcal{M}} \times \{\hat{J}\}$ inside $C^\infty(S^2, E_g) \times \mathcal{J}$ (and imposing the relevant intersection conditions).

This extension is done by an argument involving parallel transporting $\text{Obs}_u \equiv (\text{im } D_u)^\perp$ in directions orthogonal to $\ker D_u$ inside $\Omega^0(S^2, u^*TM)$ and then projecting onto $\Omega_J^{0,1}(S^2, u^*TM)$.

For small W (so we consider admissible \hat{J}' close to \hat{J}) we can ensure $\text{im } D_{u, \hat{J}'}$ and $\overline{\text{Obs}}_{u, \hat{J}'}$ are transverse inside $\Omega_{\hat{J}'}^{0,1}(S^2, u^*TM)$ and we can ensure the evaluation at z_0, z_∞ is transverse to the inclusions of F, P . This is because these conditions hold for \hat{J} . We therefore obtain a smooth parametrized moduli space

$$\mathcal{M} = \{(u, \hat{J}') \in W : \bar{\partial}_{\hat{J}'}(u) \in \text{Obs}_{u, \hat{J}'}, u(z_0) \in j_{z_0}(F), u(z_\infty) \in j_{z_\infty}(P)\}$$

and $\overline{\mathcal{M}}_{\hat{J}'}$ is obtained by compactifying the smooth subset obtained by fixing \hat{J}' . \square

8.5. Compactification of \mathcal{M} . $\mathcal{M} = \mathcal{M}_{\hat{J}} \subset \mathcal{M}_{0,2,\beta=(1,1)}(E_g)$ are curves intersecting F, P over z_0, z_∞ , which lie in $S^2 \times \mathbb{CP}^1 \subset E_g$ by the maximum principle. Simplify notation by writing $S^2 \times \mathbb{CP}^1 = \mathbb{P}^1 \times \mathbb{P}^1$, $z_0 = 0$, $z_\infty = \infty$. We may assume that $j_{z_0}F, j_{z_\infty}P$ intersect the zero section in $(0, 0)$, (∞, ∞) . Thus,

$$\begin{aligned} \mathcal{M} &= \{u : u(z) = (z, \varphi(z)), \varphi \in \text{PSL}(2, \mathbb{C}), \varphi(0) = 0, \varphi(\infty) = \infty\} \\ &= \{u : u(z) = (z, az), a \in \mathbb{C}^*\} \\ &\cong \mathbb{C}^*. \end{aligned}$$

The compactification of \mathbb{C}^* is \mathbb{P}^1 , and is obtained by considering the limits $a \rightarrow 0, a \rightarrow \infty$. For example, consider $a \rightarrow 0$. Near $(0, 0)$ the curve converges in C^∞ to $z \mapsto (z, 0)$, that is $\mathbb{P}^1 \times 0$. Near (∞, ∞) the curve can be parametrized as the locus $(\frac{1}{aw}, \frac{1}{w})$, using a local fibre coordinate $w \in \mathbb{C}$ (where $w = 0$ corresponds to ∞). So the reparametrized curve converges in C^∞ to $\infty \times \mathbb{P}^1$. Thus, $a = 0$ corresponds to

the curve $\mathbb{P}^1 \times 0$ with bubble $\infty \times \mathbb{P}^1$. Similarly, $a = \infty$ corresponds to the curve $\mathbb{P}^1 \times \infty$ with bubble $0 \times \mathbb{P}^1$. From now on, we write $\overline{\mathcal{M}} \cong \mathbb{P}^1$ for the compactification.

8.6. Description of Obs. Differential geometrically, $\text{Obs}_u = \text{coker } D_u$. We will now explain that, algebraic geometrically,

$$\text{Obs}_u = R^1 \pi_* f^* E_g$$

where $f : \mathcal{C} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is the universal curve.

Definition 50. *In our setup, the universal curve*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f=\text{ev}_3} & \mathbb{P}^1 \times \mathbb{P}^1 \\ \downarrow \pi & & \\ \overline{\mathcal{M}} \cong \mathbb{P}^1 & & \end{array}$$

is the space \mathcal{C} consisting of $u \in \overline{\mathcal{M}}$ with an additional marked point w on the domain, and f is the evaluation $f(u, w) = u(w)$. Universality is because for $u \in \mathcal{M}$, $\mathbb{P}^1 \cong \pi^{-1}(u)$ is parametrized by w and the composite $\mathbb{P}^1 \cong \pi^{-1}(u) \xrightarrow{f} \mathbb{P}^1 \times \mathbb{P}^1$ is u .

Lemma 51. $\text{Obs}_u = R^1 \pi_* f^* \mathcal{O}(-1, -1)$, where $R^1 \pi_*$ is the 1st right derived functor of the direct image functor [11, III.8]. This is the compactification for Lemma 49.

Proof. Mimick Lemma 46, but work in class $(1, 1)$ instead of $(1, 0)$. We claim that

$$u^* T^v E_g = u^* (T\mathbb{P}^1 \oplus \mathcal{O}(-1, -1)) = \mathcal{O}(2) \oplus \mathcal{O}(-2).$$

This is proved by considering the map $\phi = (\pi_g, \pi_M) \circ u : \mathbb{P}^1 \rightarrow \mathcal{O}(-1, -1) \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$. On cohomology it acts $H^2(\mathbb{P}^1 \times \mathbb{P}^1) \rightarrow H^2(\mathbb{P}^1)$ by pairing with $(1, 1)$. Finally, use that c_1 is functorial and that $T\mathbb{P}^1 = \mathcal{O}(2)$ over (the second) \mathbb{P}^1 .

$$D_u = \overline{\partial} : \Gamma(\mathbb{P}^1, \mathcal{O}(2) \oplus \mathcal{O}(-2)) \rightarrow \Omega^{0,1}(\mathbb{P}^1, \mathcal{O}(2) \oplus \mathcal{O}(-2))$$

Thus, omitting \mathbb{P}^1 references,

$$\begin{aligned} \text{Obs} &= \text{coker } \overline{\partial} = H^{0,1}(\mathcal{O}(2) \oplus \mathcal{O}(-2)) \cong H^1(\mathcal{O}(2) \oplus \mathcal{O}(-2)) \\ &\cong H^0((\mathcal{O}(2) \oplus \mathcal{O}(-2))^\vee \otimes \mathcal{O}(-2)) = H^0(\mathcal{O}(-4) \oplus \mathcal{O}) \\ &= H^0(\mathbb{P}^1, \mathcal{O}) \quad (\text{complex 1 dimensional.}) \end{aligned}$$

So only the $\mathcal{O}(-1, -1)$ contributes to Obs. By universality, the stalk is

$$\text{Obs}_u = H^1(\mathbb{P}^1, u^* \mathcal{O}(-1, -1)) \cong H^1(\pi^{-1}(u), f^* \mathcal{O}(-1, -1)) = (R^1 \pi_* f^* \mathcal{O}(-1, -1))_u$$

which shows that the map $\text{Obs} \rightarrow R^1 \pi_* f^* \mathcal{O}(-1, -1)$ (obtained similarly) is an isomorphism of sheaves. Since $R^1 \pi_* f^* \mathcal{O}(-1, -1)$ makes sense also over the compactification, we may take that as the definition of $\overline{\text{Obs}}$ in Lemma 49. \square

Lemma 52. $f : \mathcal{C} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at $(0, 0)$ and (∞, ∞) .

Proof. Consider $Q = f^{-1}(z_3, y_3)$. If $z_3 \neq 0, \infty$ and $y_3 \neq 0, \infty$, then Q is a unique point in \mathcal{C} corresponding to a curve (with additional marked point (z_3, y_3)).

For $(z_3 = \infty, y_3 \neq \infty)$ and $(z_3 \neq 0, y_3 = 0)$, Q is a point corresponding to $a = 0$. For $(z_3 \neq \infty, y_3 = \infty)$ and $(z_3 = 0, y_3 \neq 0)$, Q is a point corresponding to $a = \infty$.

On the other hand, $f^{-1}(0, 0) \cong \mathbb{P}^1$, $f^{-1}(\infty, \infty) \cong \mathbb{P}^1$ corresponding to all $a \in \mathbb{P}^1$ (with additional marked point at $(0, 0)$ and (∞, ∞) respectively).

So f is a biholomorphism except over $(0, 0)$, (∞, ∞) . One could argue that since f is a birational morphism of algebraic surfaces it must be a composite of blow-ups. Alternatively, explicitly near $(0, 0)$ (the case (∞, ∞) is similar) we have a

parametrization for \mathcal{C} given by $((z_3, y_3), [Z_3 : Y_3]) \in \mathbb{C} \times \mathbb{C}P^1$ satisfying $z_3 Y_3 = Z_3 y_3$, corresponding to $a = Y_3/Z_3$ with additional marked point (z_3, y_3) . \square

Theorem 53. $\text{Obs} = R^1\pi_* f^*\mathcal{O}(-1, -1) \rightarrow \overline{\mathcal{M}}$ is isomorphic to $\mathcal{O}(1) \rightarrow \mathbb{P}^1$, so

$$A = \langle e(\text{Obs}), \overline{\mathcal{M}} \rangle = \text{degree}(\mathcal{O}(1)) = 1.$$

Proof. Let $\mathcal{F} = f^*\mathcal{O}(-1, -1)$.

Sub-claim.

$$1 - \deg(R^1\pi_*\mathcal{F}) = \int_{\mathcal{C}} \text{ch}(\mathcal{F}) \text{td}(TC).$$

Proof. Recall the direct image in K -theory [11, Appendix A] for a proper morphism $g : X \rightarrow Y$ is $g_! = \sum (-1)^i R^i g_* : K(X) \rightarrow K(Y)$.

For $g : \mathbb{P}^1 \rightarrow \text{point}$ and a vector bundle \mathcal{G} on \mathbb{P}^1 , by Riemann-Roch:

$$\text{rank}_{\mathbb{C}} \mathcal{G} + \deg(\mathcal{G}) = \chi_{\text{holo}}(\mathcal{G}) = h^0(\mathbb{P}^1, \mathcal{G}) - h^1(\mathbb{P}^1, \mathcal{G}) = h^0(\text{point}, g_!\mathcal{G}).$$

Consider the composite $\mathcal{C} \xrightarrow{\pi} \mathbb{P}^1 \xrightarrow{g} \text{point}$. Since $(g\pi)_* = g_*\pi_*$ also $(g\pi)_! = g_!\pi_!$, so:

$$1 + \deg(\pi_!\mathcal{F}) = h^0((g\pi)_!\mathcal{F}).$$

Grothendieck-Riemann-Roch (see Fulton [9, Sec.15.2]), written in K -theory, states:

$$(g\pi)_!(\mathcal{F}) \cdot \text{td}(\text{point}) = (g\pi)_*(\text{ch}(\mathcal{F}) \cdot \text{td}(\mathcal{C})).$$

So, using $\text{td}(\text{point}) = 1$, and taking h^0 , get: $1 + \deg(\pi_!\mathcal{F}) = \int_{\mathcal{C}} \text{ch}(\mathcal{F}) \wedge \text{td}(\mathcal{C})$ where we switched to cohomology notation on the right hand side (intersection product of complementary cycles is integration of the wedge product of the Poincaré dual cocycles, and we used that push-forward of a point is a point).

Finally, for dimensional reasons, $\pi_!\mathcal{F} = R^0\pi_*\mathcal{F} - R^1\pi_*\mathcal{F}$. Moreover, the R^0 term vanishes since it has stalk $H^0(\mathbb{P}^1, u^*\mathcal{O}(-1, -1)) = H^0(\mathbb{P}^1, \mathcal{O}(-2)) = 0$ (geometrically: you cannot deform sections away from the zero section by the maximum principle). This proves the Sub-claim. \checkmark

In our case, the Todd class is

$$\text{td}(\mathcal{C}) \equiv \text{td}(TC) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) \in H^*(\mathcal{C}, \mathbb{Z}) \otimes \mathbb{Q},$$

where we abbreviate $c_i = c_i(TC)$, and the Chern character is just

$$\text{ch}(\mathcal{F}) = e^z \in H^*(\mathcal{C}, \mathbb{Z}) \otimes \mathbb{Q},$$

where $z = c_1(\mathcal{F}) = f^*c_1(\mathcal{O}(-1, -1))$.

Now we calculate the integral in the sub-claim, which expands to:

$$\frac{1}{12} \int_{\mathcal{C}} c_2 + \frac{1}{12} \int_{\mathcal{C}} c_1^2 + \frac{1}{2} \int_{\mathcal{C}} c_1 z + \frac{1}{2} \int_{\mathcal{C}} z^2$$

The first integral is the Euler characteristic of \mathcal{C} , which is 6, since \mathcal{C} has Betti numbers 1, 0, 4, 0, 1 (the homology of $\mathbb{P}^1 \times \mathbb{P}^1$ with two additional exceptional \mathbb{P}^1).

Recall the following four facts [2, Prop II.3] about intersection products of divisors in a blow-up $\pi : R \rightarrow S$ of algebraic surfaces at a point with exceptional divisor E : $\pi^*D \cdot \pi^*D' = D \cdot D'$, $E \cdot \pi^*D = 0$, $E \cdot E = -1$, $K_R = \pi^*K_S + E$ (where K_S is the canonical divisor class corresponding to T^*S).

The last fact implies: $T^*\mathcal{C} = f^*T^*(\mathbb{P}^1 \times \mathbb{P}^1) + (E_1 + E_2)$ (in K -theory), where E_1, E_2 are the two exceptional fibres of f . Thus, by the other three facts, and because E_1, E_2 don't intersect:

$$TC^2 = T(\mathbb{P}^1 \times \mathbb{P}^1)^2 - 2 = \langle (2, 2), (2, 2) \rangle - 2 = 6,$$

so the second integral $\int_{\mathcal{C}} c_1^2 = 6$.

By the second fact, working in K -theory, the third integral is:

$$\begin{aligned} c_1 \cdot z &= (f^*T(\mathbb{P}^1 \times \mathbb{P}^1) + (E_1 + E_2)) \cdot f^*\mathcal{O}(-1, -1) \\ &= T(\mathbb{P}^1 \times \mathbb{P}^1) \cdot \mathcal{O}(-1, -1) \\ &= \langle (2, 2), (-1, -1) \rangle = -4 \end{aligned}$$

The last integral:

$$f^*\mathcal{O}(-1, -1) \cdot f^*\mathcal{O}(-1, -1) = \mathcal{O}(-1, -1)^2 = \langle (-1, -1), (-1, -1) \rangle = 2.$$

Therefore:

$$1 - \deg(\text{Obs}) = \int_{\mathcal{C}} \text{ch}(\mathcal{F}) \wedge \text{td}(\mathcal{C}) = \frac{6}{12} + \frac{6}{12} - \frac{4}{2} + \frac{2}{2} = 0.$$

Thus $\deg(\text{Obs}) = 1$, and line bundles over \mathbb{P}^1 are classified by their degree. \square

8.7. Symplectic cohomology of $\mathcal{O}(-1) \rightarrow \mathbb{CP}^1$.

Theorem 54. *Let M be the total space of $\mathcal{O}(-1) \rightarrow \mathbb{CP}^1$. Then $SH^*(M) \cong \Lambda \cdot 1$, and $c^* : QH^*(M) \rightarrow SH^*(M)$ maps $c^*(1) = 1$, $c^*(\omega_{\mathbb{CP}^1}) = -t \cdot 1$.*

Proof. Combining Theorem 53 with Lemmas 49 and 48 we obtain

$$r_{\tilde{g}} = \begin{bmatrix} t & -1 \\ 0 & 0 \end{bmatrix} : \Lambda^2 \rightarrow \Lambda^2.$$

So by Theorem 21, $SH^*(M) \cong \Lambda \cdot 1$, where $1 = \psi^-(1) \in SH^{\text{even}}(M)$ is the unit. Recall $\omega_{\mathbb{CP}^1} = \text{PD}([F])$, $1 = \text{PD}([M])$, so $r_{\tilde{g}}(\omega_{\mathbb{CP}^1}) = t\omega_{\mathbb{CP}^1}$ and $r_{\tilde{g}}(1) = -\omega_{\mathbb{CP}^1} = c_1(\mathcal{O}(-1))$. This represents the continuation $SH^*(H_0) \rightarrow SH^*(H_1)$, after identifications with $QH^*(M)$, and this in turn is identified with c^* yielding:

$$SH^*(M) = QH^*(M) / \ker r_{\tilde{g}} = \Lambda[\omega_{\mathbb{CP}^1}] / (\omega_{\mathbb{CP}^1} + t \cdot 1). \quad \square$$

9. SYMPLECTIC COHOMOLOGY OF $M = \text{Tot}(\mathcal{O}(-n) \rightarrow \mathbb{P}^m)$

9.1. Description of $\mathbf{M} = \text{Tot}(\mathcal{O}(-\mathbf{n}) \rightarrow \mathbb{P}^{\mathbf{m}})$. Let $M = \text{Tot}(\mathcal{O}(-n) \rightarrow \mathbb{P}^m)$.

From now on, we always use complex dimensions to avoid factors of 2 everywhere. $H^*(\mathbb{P}^m)$ is generated by $\omega^m, \omega^{m-1}, \dots, \omega, 1$. Here $\omega = \pi_M^* \omega_{\mathbb{P}^1}$ and $\omega_{\mathbb{P}^1}[\mathbb{P}^1 \subset \mathbb{P}^m] = 1$. Poincaré dually, $H_*^{lf}(\mathbb{P}^m)$ is generated by lf cycles $F_1, F_2, \dots, F_m, F_{m+1} = M$ where $F_j = \pi_M^{-1}(\mathbb{P}^{j-1})$ for some equatorial $\mathbb{P}^1 \subset \mathbb{P}^2 \subset \dots \subset \mathbb{P}^{m-1} \subset \mathbb{P}^m$, and $j = \dim_{\mathbb{C}} F_j$.

These lf cycles are dual, with respect to the intersection product, to the cycles $\mathbb{P}^m, \mathbb{P}^{m-1}, \dots, \mathbb{P}^1, pt = \mathbb{P}^0$ since $\mathbb{P}^{1+m-j} \bullet \mathbb{P}^{j-1} = 1$ in \mathbb{P}^m . The condition of sweeping out F_j at z_{∞} is thus equivalent to the intersection condition over z_{∞} with its dual: the (perturbed) \mathbb{P}^{1+m-j} . For genericity, one needs to perturb: for $0 < j \leq m$, the cycle \mathbb{P}^j can be perturbed vertically (in the smooth category) to a cycle which intersects the zero section in $-n[\mathbb{P}^{j-1}]$, which is the Poincaré dual of the Euler class of $\mathcal{O}(-n)$ ($\mathcal{O}(-n)$ pulls back to $\mathcal{O}(-n) \rightarrow \mathbb{P}^j$ via $\mathbb{P}^j \hookrightarrow \mathbb{P}^m$).

This time, $c_1(TM)[\mathbb{P}^1] = c_1(T\mathbb{P}^m)[\mathbb{P}^1] + c_1(\mathcal{O}(-n))[\mathbb{P}^1] = 1 + m - n$. So define

$$\boxed{N = 1 + m - n}$$

As before $\Lambda = \mathbb{Z}[t^{-1}, t]$ as $\pi_2(M)$ is generated by $t = [\mathbb{P}^1]$, and $|t| = -2N$ (homological grading). So weak⁺ monotonicity holds except in a small range:

- (1) $1 \leq \mathbf{n} < 1 + \mathbf{m}$: M is monotone \checkmark ($c_1(TM)$ is a positive multiple of ω_M);
- (2) $\mathbf{n} = 1 + \mathbf{m}$: critical case: $c_1(TM) = 0$ \checkmark (so $SH^*(M) = 0$ by Theorem 6);
- (3) $2 + \mathbf{m} \leq \mathbf{n} \leq 2\mathbf{m}$: this is the range where weak⁺-monotonicity fails. There may be technical issues in constructing $r_{\tilde{g}}$ so we will not discuss this;

(4) $\mathbf{n} \geq \mathbf{1} + \mathbf{2m}$: $|N| \geq \dim_{\mathbb{C}} \mathbb{P}^m = m$ ✓ (and $SH^*(M) = 0$ by Corollary 56).

The space of (j, \hat{J}) -holomorphic sections has complex dimension

$$\text{vir} \dim_{\mathbb{C}} \mathcal{S}(t^d + S_{\hat{g}}) = \dim_{\mathbb{C}} M + c_1(T E_g^v)(S_{\hat{g}}) + dc_1(TM)(t) = 1 + m - 1 + dN = m + Nd.$$

The intersection condition at z_0 with F_j cuts this down by $1 + m - j$. Therefore,

$$\begin{aligned} \text{vir} \dim_{\mathbb{C}} (\mathcal{S}(t^d + S_{\hat{g}}) \cap \text{ev}_{z_0}^{-1}(F_j) \cap \text{ev}_{z_{\infty}}^{-1}(\mathbb{P}^{1+m-i})) &= m + Nd - (1 + m - j) - i \\ &= Nd - i + j - 1. \end{aligned}$$

So provided $\boxed{Nd - i + j - 1 = 0}$ this contributes to the entry (i, j) of the matrix $r_{\hat{g}}$ viewed as an $(m+1) \times (m+1)$ matrix over Λ in the basis F_1, F_2, \dots, F_{m+1} (or cohomologically: in the basis $\omega^m, \omega^{m-1}, \dots, 1$).

Lemma 55. *The constants are always regular for $\hat{J} = \begin{bmatrix} j & 0 \\ 0 & j \end{bmatrix}$, J integrable, and $r_{\hat{g}}$ has the following form in the basis $\omega^m, \omega^{m-1}, \dots, \omega, 1$:*

$$r_{\hat{g}} = \begin{bmatrix} 0 & -n & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & 0 & -n & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_0 t & 0 & \cdots & 0 & -n & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & A_1 t & \cdots & 0 & 0 & -n & 0 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ B_0 t^2 & 0 & \cdots & 0 & 0 & 0 & 0 & -n & 0 & \cdots & \cdots \\ 0 & B_1 t^2 & \cdots & 0 & 0 & 0 & 0 & 0 & -n & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -n \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The $-n = c_1(\mathcal{O}(-n))[\mathbb{P}^1]$ arise on the second main diagonal, they count constant sections. The A_0, B_0, C_0, \dots in positions $(N, 1), (2N, 1), (3N, 1), \dots$ and the corresponding subdiagonals with entries A_a, B_a, C_a, \dots count sections in class $\beta = (1, 1), (1, 2), (1, 3), \dots$. All other entries are zero. Moreover:

$$\begin{aligned} A_a &= \text{GW}_{0,2,(1,1)}^{E_g}(j_{z_0} F_{a+1}, j_{z_{\infty}} \mathbb{P}^{1+m-a-N}) = \text{GW}_{0,2,(1,1)}^{E_g}(j_{z_0} F_{a+1}, j_{z_{\infty}} \mathbb{P}^{n-a}) \\ B_a &= \text{GW}_{0,2,(1,2)}^{E_g}(j_{z_0} F_{a+1}, j_{z_{\infty}} \mathbb{P}^{1+m-a-2N}) \\ C_a &= \text{GW}_{0,2,(1,3)}^{E_g}(j_{z_0} F_{a+1}, j_{z_{\infty}} \mathbb{P}^{1+m-a-3N}) \\ &\dots \end{aligned}$$

where $j_{z_0}, j_{z_{\infty}} : M \rightarrow E_g$ are the inclusions of the fibres over $z_0, z_{\infty} \in \mathbb{P}^1$.

Proof. For regularity of constants see Theorem 64. For $d = 0$, the constants sweep out the 1f cycle $[\mathbb{P}^m]$ under $\text{ev}_{z_{\infty}}$. So the contribution to $r_{\hat{g}}(F_j)$ is $[\mathbb{P}^m] \cap F_j = [\mathbb{P}^{j-1}]$. Intersecting with a perturbed \mathbb{P}^{1+m-i} , where $i = j - 1$, is $-n[\text{pt}]$ (the perturbation hits the zero section in $-n[\mathbb{P}^{m-i}]$). So constants contribute $-nF_{j-1}$ to $r_{\hat{g}}(F_j)$. The last row vanishes because it involves an intersection condition with a point, which we can move to infinity (without affecting $r_{\hat{g}}(1)$ in cohomology), so the moduli spaces will never intersect it by the maximum principle. The rest is by dimensions. \square

Corollary 56. *For $n > 2m$, $\text{vir} \dim_{\mathbb{C}} \mathcal{S}(t^d + S_{\hat{g}}) = m + Nd < m - md$ so only $d = 0$ occurs, so $r_{\hat{g}}$ only has a supdiagonal of $-n$'s, so $r_{\hat{g}}$ is nilpotent, so $SH^*(M) = 0$.*

Arguing as in Lemma 51, for u in class $(1, d)$,

$$u^* T^v E_g = u^*(T\mathbb{P}^m \oplus \mathcal{O}(-1, -n)) = \mathcal{O}(2d) \oplus \mathcal{O}(d) \oplus \cdots \oplus \mathcal{O}(d) \oplus \mathcal{O}(-1 - nd)$$

with $m - 1$ copies of $\mathcal{O}(d)$. Here we used that fact that $\mathbb{P}^1 \subset \mathbb{P}^m$ has tangent bundle $\mathcal{O}(2)$ and normal bundle $\nu_{\mathbb{P}^1 \subset \mathbb{P}^m} = \nu_{\mathbb{P}^1 \subset \mathbb{P}^2} \oplus \nu_{\mathbb{P}^2 \subset \mathbb{P}^3} \oplus \cdots \oplus \nu_{\mathbb{P}^{m-1} \subset \mathbb{P}^m}$, and $c_1(\nu_{\mathbb{P}^{j-1} \subset \mathbb{P}^j}) = c_1(T\mathbb{P}^j|_{\mathbb{P}^{j-1}}) - c_1(T\mathbb{P}^{j-1}) = 1 \cdot \omega_{\mathbb{P}^1}$. Thus:

$$\begin{aligned} \ker \bar{\partial} &= H^0(\mathbb{P}^1, \mathcal{O}(2d) \oplus \mathcal{O}(d)^{\oplus m-1} \oplus \mathcal{O}(-1 - nd)) \\ &= H^0(\mathbb{P}^1, \mathcal{O}(2d) \oplus \mathcal{O}(d)^{\oplus m-1}) \\ \text{coker } \bar{\partial} &= H^1(\mathbb{P}^1, \mathcal{O}(2d) \oplus \mathcal{O}(d)^{\oplus m-1} \oplus \mathcal{O}(-1 - nd)) \\ &\cong H^0(\mathbb{P}^1, \mathcal{O}(-2d - 2) \oplus \mathcal{O}(-d - 2)^{\oplus m-1} \oplus \mathcal{O}(nd - 1))^\vee \\ &\cong H^0(\mathbb{P}^1, \mathcal{O}(nd - 1))^\vee \end{aligned}$$

So the obstruction bundle Obs has $\text{rank}_{\mathbb{C}} = nd$. To determine $r_{\tilde{g}}$, all $0 \leq d \leq \frac{m}{1+m-n}$ will contribute for $n < 1 + m$. The A_a, B_a, \dots are in principle determined by $\langle e(\text{Obs}), [\mathcal{M}] \rangle$ where \mathcal{M} is the (compactified) moduli space of sections cut down by the relevant intersection conditions described before the Lemma. In practice Obs becomes rapidly unwieldy for $n \neq 1, d > 1$. We now study $n = 1$ explicitly.

9.2. Explicit description for $\mathbf{M} = \text{Tot}(\mathcal{O}(-1) \rightarrow \mathbb{P}^m)$.

Lemma 57. For $M = \text{Tot}(\mathcal{O}(-1) \rightarrow \mathbb{P}^m)$,

$$r_{\tilde{g}} = \begin{bmatrix} 0 & -1 & 0 & 0 & \cdots \\ 0 & 0 & -1 & 0 & \cdots \\ \vdots & & & & \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 \\ t & 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$SH^*(M) = \Lambda[\omega_Q]/(\omega_Q^m + t \cdot 1)$$

and $c^* : QH^*(M) \rightarrow SH^*(M)$ maps $c^*(1) = 1$, $c^*(\omega_Q) = \omega_Q$, $c^*(\omega_Q^m) = -t \cdot 1$.

Proof. We only need to find the entry A_0 . This involves $d = 1$, and intersection conditions over z_0 with the fibre F_1 and over z_∞ with \mathbb{P}^1 . Perturbing \mathbb{P}^1 vertically, it will intersect the zero section in $-pt$. The holomorphic sections of E_g lie in the zero section, and we want those in class $(1, 1)$ which intersect $(0, 0), (\infty, \infty)$ (where in the second entry, we can assume that $0, \infty \in \mathbb{P}^1 \subset \mathbb{P}^m$ are the intersections of F and $(\mathbb{P}^1$ perturbed) with the zero section). So we reduce to maps

$$\mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^1 \times \mathbb{P}^m,$$

where the first maps are the same as in 8.6, and the second map is the inclusion. That inclusion pulls back $\mathcal{O}(-1, -1)$ to $\mathcal{O}(-1, -1)$, so the same Grothendieck-Riemann-Roch argument proves $A_0 = 1$. The rest follows as in Theorem 54. \square

Theorem 58. $QH^*(M) = \Lambda[\omega_Q]/(\omega_Q^{m+1} + t \cdot \omega_Q)$ for $\mathcal{O}(-1) \rightarrow \mathbb{P}^m$.

Proof. Denote ω the canonical generator of $H^2(\mathbb{P}^m)$. We denote ω^k the ordinary cup product powers, and ω_Q^k the quantum cup product powers.

For $\mathcal{O}(-n) \rightarrow \mathbb{P}^m$ we first calculate for each $j = 1, \dots, m$:

$$\omega * \omega^j = \sum_{\ell=1+j-dN} \text{GW}_{0,3,d}^M(F_m, F_{m+1-j}, \mathbb{P}^\ell) \cdot t^d \cdot \omega^\ell$$

where we used that $\omega^\ell = \text{PD}(F_{m+1-\ell})$ and $\mathbb{P}^\ell = \text{D}(F_{m+1-\ell})$ (where PD is Poincaré duality and D is intersection duality), and we used the (complex) GW dimension condition $(1) + (j) + (1 + m - \ell) = (1 + m) + Nd + 3 - 3$.

For $\mathcal{O}(-1) \rightarrow \mathbb{P}^m$ we have $N = 1 + m - n = m$, and since $1 \leq j \leq m$ we have $0 \leq \ell = 1 + j - dN \leq 1 + m - dm$, so $d = 0$ or 1 . For $d = 0$ we count constant $\mathbb{P}^1 \rightarrow M$, so the 1f cycle we get under evaluation is M and the GW invariant counts

$$M \cap F_m \cap F_{m+1-j} \cap \mathbb{P}^\ell = \mathbb{P}^{m-1} \cap \mathbb{P}^{m-j} \cap \mathbb{P}^\ell = \mathbb{P}^{\ell-1-j} = \mathbb{P}^0$$

so we get the ordinary cup product contributions $\omega * \omega^j = \omega^{1+j} + \dots$. The case $d = 1$ forces $j = m - 1$ or $j = m$. For $j = m - 1$: \mathbb{P}^0 can be moved to infinity so $\text{GW} = 0$. Finally consider $j = m$, $\ell = 1$. Regularity of degree $d = 1$ holomorphic $u : \mathbb{P}^1 \rightarrow \mathbb{P}^m \subset M$ follows from $u^*TM = u^*T\mathbb{P}^m \oplus \mathcal{O}(-1)$, $u^*T\mathbb{P}^m \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus m-1}$,

$$\begin{aligned} \text{coker } \bar{\partial} &\cong H^1(\mathbb{P}^1, \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus m-1} \oplus \mathcal{O}(-1)) \\ &\cong H^0(\mathbb{P}^1, \mathcal{O}(-4) \oplus \mathcal{O}(-3)^{\oplus m-1} \oplus \mathcal{O}(-1))^\vee = 0. \end{aligned}$$

For $d = 1, j = m$, we impose intersection conditions F_m, F_1, \mathbb{P}^1 . Perturbing that \mathbb{P}^1 off the zero section, these three conditions inside the zero section become conditions $\mathbb{P}^{m-1}, \mathbb{P}^0, -1 \cdot \text{pt.}$ There is a unique holomorphic \mathbb{P}^1 through two points, and it automatically intersects the \mathbb{P}^{m-1} , so $\text{GW}_{0,3,1}^M(F_m, F_1, \mathbb{P}^1) = -1$.

Conclusion: $\omega * \omega^j = \omega^{1+j}$ for $j = 1, \dots, m - 1$, so $\omega_Q^{1+j} = \omega^{1+j}$, and

$$\omega_Q^{1+m} = \omega * \omega_Q^m = \omega * \omega^m = \omega^{1+m} - t\omega = -t\omega.$$

So $r_{\tilde{g}}(1) * \omega^n = (-\omega) * \omega^n = t\omega$ confirming Lemma 57 via Theorem 1. \square

9.3. Quantum cohomology of $M = \text{Tot}(\mathcal{O}(-n) \rightarrow \mathbb{P}^m)$.

Corollary. *Quantum cup product by $c_1(\mathcal{O}(-n)) = -n\omega$ defines the matrix r_g of Lemma 55 in the basis $\omega^m, \dots, \omega, 1$, and so*

$$\begin{aligned} A_a &= -n \cdot \text{GW}_{0,3,1}^M(F_m, F_{a+1}, \mathbb{P}^{1+m-a-N}) = -n \cdot \text{GW}_{0,3,1}^M(F_m, F_{a+1}, \mathbb{P}^{n-a}) \\ B_a &= -n \cdot \text{GW}_{0,3,2}^M(F_m, F_{a+1}, \mathbb{P}^{1+m-a-2N}) \\ C_a &= -n \cdot \text{GW}_{0,3,3}^M(F_m, F_{a+1}, \mathbb{P}^{1+m-a-3N}) \\ &\dots \end{aligned}$$

Remark. *The obstruction bundle involved in calculating the A_a, B_a, C_a, \dots in this way has fiber $H^0(\mathbb{P}^1, \mathcal{O}(nd - 2))^\vee$ of (complex) rank $nd - 1$.*

9.4. Linear algebra. Let $M = \text{Tot}(\mathcal{O}(-n) \rightarrow \mathbb{P}^m)$ (although what we say applies also to the cyclic subgroups of QH^*, SH^* generated by $c_1(L)$ for $M = \text{Tot}(L \rightarrow B)$).

Let $c = c_1(\mathcal{O}(-n)) = -n\omega_Q$. Taking quantum cup product powers of c yields $c_Q^m, c_Q^{m-1}, \dots, c_Q, 1$, which is a basis for $QH^*(M)$ in characteristic 0 (and for odd n in characteristic 2). The $r_{\tilde{g}}$ in this basis turns into the canonical form:

$$\begin{bmatrix} -a_1 & 1 & 0 & 0 & \dots \\ -a_2 & 0 & 1 & 0 & \dots \\ \vdots & & & & \\ -a_m & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & & 0 \end{bmatrix}$$

where $\lambda^{m+1} + a_1\lambda^m + a_2\lambda^{m-1} + \dots + a_m\lambda$ is the characteristic polynomial of $r_{\tilde{g}}$. Here $a_i = 0$ if i is not divisible by $|N|$, and a_i is homogeneous in t of order $t^{N/i}$.

Since $r_{\tilde{g}}$ is quantum cup product by c ,

$$QH^*(M) \equiv \Lambda[c_Q]/(c_Q^{m+1} + a_1c_Q^m + \dots + a_m c_Q).$$

Suppose there is a largest integer $m \geq p \geq 1$ for which $a_p \neq 0$ (otherwise $c_Q^{m+1} = 0$ and $SH^*(M) = 0$). Then the characteristic polynomial of $r_{\tilde{g}}$ is

$$\lambda^{m+1-p}(\lambda^p + a_1\lambda^{p-1} + \dots + a_p).$$

Since $\text{rank } r_{\tilde{g}} = m$, the above implies the Jordan normal form of $r_{\tilde{g}}$ has exactly one Jordan block for eigenvalue 0 of size $m + 1 - p$. Thus, for $k \geq m + 1 - p$, $\ker r_{\tilde{g}}^k$ is the generalized eigenspace of $r_{\tilde{g}}$ for eigenvalue 0 which is

$$\ker r_{\tilde{g}}^k = \Lambda \cdot (\lambda^p + a_1 \lambda^{p-1} + \cdots + a_p)$$

Remark: $\text{image}(r_{\tilde{g}}^k) = \text{span}(c_Q^m, \dots, c_Q^{1+m-p})$ stabilizes for $k \geq m + 1 - p$.

It follows by Theorem 1 that $SH^*(M)$ has rank p since

$$SH^*(M) \cong \Lambda[c_Q]/(c_Q^p + a_1 c_Q^{p-1} + \cdots + a_p).$$

Lemma 59. $a_N = (-1)^N n^{N-1} \sum_{j=0}^{n-1} A_j t$ (where $N = 1 + m - n \geq 0$).

Proof. If the matrix in Lemma 55 had $-n$'s replaced by -1 and A_j, B_j, \dots replaced by $\tilde{A}_j, \tilde{B}_j, \dots$, then one can easily check that the characteristic polynomial would be $\lambda^{m+1} + \tilde{a}_N \lambda^{m+1-N} + \cdots$ where $\tilde{a}_N = (-1)^N \sum \tilde{A}_j t$. If we replace $r_{\tilde{g}}$ by $r_{\tilde{g}}/n$ the matrix has that form with $\tilde{A}_j = A_j/n$. Under this replacement, the characteristic polynomial changes from $\lambda^{1+m} + a_N \lambda^{1+m-N} + \cdots$ to $\lambda^{1+m} + a_N n^{-N} \lambda^{1+m-N} + \cdots$. So $a_N n^{-N} = \tilde{a}_N = (-1)^N \sum A_j t/n$. \square

Corollary 60. For $2N > m$ (equivalently $n < 1 + \frac{m}{2}$) only the A_j contribute to $r_{\tilde{g}}$, and the only non-zero a_i is $a_N = -(-n)^{N-1} \sum A_j t$, so putting $\alpha = \sum A_j$:

$$QH^*(M) = \Lambda[c_Q]/(c_Q^{1+m} - (-n)^{N-1} \alpha t c_Q^N) = \Lambda[\omega_Q]/(\omega_Q^{1+m} + n^{-1} \alpha t \omega_Q^N)$$

$$SH^*(M) = \Lambda[c_Q]/(c_Q^N - (-n)^{N-1} \alpha t) = \Lambda[\omega_Q]/(\omega_Q^N + n^{-1} \alpha t)$$

where $N = 1 + m - n$, and in Theorem 63 we calculate A_j .

9.5. Calculation of A_a by virtual localization. We follow closely the notation of Pandharipande's notes [17], which are based on Graber-Pandharipande [10]. Localization was first applied to stable maps by Kontsevich [13]. We also mention Cox-Katz [6, p.277] as a good reference. As a warm-up we redo the $\mathcal{O}(-1) \rightarrow \mathbb{P}^1$.

Theorem 61. For $\mathcal{O}(-1) \rightarrow \mathbb{P}^1$, $A_0 = 1$.

Proof. Consider the deformation long exact sequence [17, p.549],

$$0 \rightarrow \text{Aut}(C) \rightarrow \text{Def}(u) \rightarrow \text{Def}(C, u) \rightarrow \text{Def}(C) \rightarrow \text{Ob}(u) \rightarrow \text{Ob}(C, u) \rightarrow 0$$

where $C = (\Sigma, x_1, x_2)$ is a 2-pointed nodal curve of arithmetic genus 0, and $u : C \rightarrow E_g$ are the sections in class $(1, 1)$ that we want to count in Lemma 48. The following observations clarify how our setup is different from [17]:

- (1) The marked points x_1, x_2 are fixed in our setup, indeed as in 8.5 we choose $x_1 = p_{00} = (0, 0)$, $x_2 = p_{11} = (\infty, \infty)$.
- (2) The holomorphic maps we consider are

$$u : C \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \subset E_g$$

in class $(1, 1)$, having already imposed the intersection conditions F, P - so we use the moduli space $\overline{\mathcal{M}}$ of 8.4. We will often refer to the second \mathbb{P}^1 as $\mathbb{CP}^1 \subset M$ to distinguish it from the first factor.

- (3) The open part \mathcal{M} are maps of the form $u(z) = (z, az)$. The compactification gives rise to two new stable maps $U_{10}, U_{01} : \Sigma_1 \cup \Sigma_2 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$, where C is a nodal curve with two \mathbb{P}^1 's joined at one node v . The first map is specified by: $U_{10}(\Sigma_1) = \mathbb{P}^1 \times 0$, $U_{10}(\Sigma_2) = \infty \times \mathbb{P}^1$, $U_{10}(v) = p_{10} = (\infty, 0)$. The second: $U_{01}(\Sigma_1) = 0 \times \mathbb{P}^1$, $U_{01}(\Sigma_2) = \mathbb{P}^1 \times \infty$, $U_{01}(v) = p_{01} = (0, \infty)$.

- (4) The torus action by $\mathbb{T} = (\mathbb{C}^*)^2 \ni t$ on $\mathbb{P}^1 \times \mathbb{P}^1$ is:

$$([z_0 : z_1], [w_0 : w_1]) \mapsto ([z_0 : z_1], [t_0^{-1} w_0 : t_1^{-1} w_1]).$$

(the inverses ensure that the action on linear forms in $H^0(\mathbb{P}^1, \mathcal{O}(1))$ involves no inverses). This induces a natural action on $u(z) = ([z_0 : z_1], [z_0 : az_1])$:

$$(t \cdot u)(z) = ([z_0 : z_1], [t_0^{-1} z_0 : at_1^{-1} z_1]).$$

Denote α_0, α_1 the weights for t .

- (5) The \mathbb{T} -fixed points of $\overline{\mathcal{M}}$ are the two maps U_{01}, U_{10} . We call Γ_{10}, Γ_{01} the decorated graphs which describe U_{10}, U_{01} (explicitly: graphs with two edges, and vertices labeled by 00, 10, 11 and 00, 01, 11 respectively).
- (6) Because of the intersection conditions, we only consider deformations of u subject to the conditions $u(0) = p_{00}, u(\infty) = p_{11}$. There are no reparametrization automorphisms on the main component of u because we only consider sections. There are $PSL(2, \mathbb{C})$ -reparametrization automorphisms for the bubbles arising in the M -fibres of E_g .
- (7) E_g plays the same role as \mathbb{P}^m in [17, 27.6], however we do not consider deformations of u in all TE_g -directions, but rather only in $T^v E_g$ -directions since we are working with sections. Recall $T^v E_g \cong T\mathbb{CP}^1 \oplus \mathcal{O}(-1, -1)$. So

$$\begin{aligned} (U_{10}^* T^v E_g)|_{\Sigma_1} &\cong T_0 \mathbb{CP}^1 \oplus \mathcal{O}(-1), & (U_{10}^* T^v E_g)|_{\Sigma_2} &= TM \cong T\mathbb{CP}^1 \oplus \mathcal{O}(-1), \\ (U_{01}^* T^v E_g)|_{\Sigma_2} &\cong T_\infty \mathbb{CP}^1 \oplus \mathcal{O}(-1), & (U_{01}^* T^v E_g)|_{\Sigma_1} &= TM \cong T\mathbb{CP}^1 \oplus \mathcal{O}(-1) \end{aligned}$$

We use the convention of [17] that we refer to the fiber of a vector bundle when we mean the vector bundle. In our setup, $\text{Ob}(C, u) = 0$ since there are no contracted components in our stable maps. The obstruction bundle is $\text{Ob}(u) = H^1(C, u^* T^v E_g)$, but the deformation bundle $\text{Def}(u)$ is not all of $H^0(C, u^* T^v E_g)$ because of the intersection conditions.

By (6), $\text{Def}(u)^{\text{mov}} = 0$ (the section of $\mathcal{O}(2)$ vanishing at $0, \infty$ has weight zero, so contributes to $\text{Def}(u)^{\text{fix}}$ and it cancels with the bubble reparametrization automorphisms $\text{Aut}(C)^{\text{fix}}$ in the deformation LES). Also by (6): $\text{Aut}(C)^{\text{mov}} = 0$.

By the Atiyah-Bott localization theorem, we want to calculate:

$$A_0 = \int_{\overline{\mathcal{M}}} e(\text{Obs}) = \int_{\overline{\mathcal{M}}^{\text{vir}}} 1 = i_{\text{point}}^* \int_{\overline{\mathcal{M}}_{\mathbb{T}}} 1 = \sum_{\Gamma} \frac{1}{e^{\mathbb{T}}(N_{\Gamma}^{\text{vir}})}$$

where we sum over our two graphs $\Gamma = \Gamma_{10}$ and Γ_{01} , and where the equivariant Euler class of the virtual normal bundle to the fixed points U_{10}, U_{01} is:

$$e^{\mathbb{T}}(N_{\Gamma}^{\text{vir}}) = \frac{e(\text{Def}(u)^{\text{mov}}) e(\text{Def}(C)^{\text{mov}})}{e(\text{Ob}(u)^{\text{mov}}) e(\text{Aut}(C)^{\text{mov}})} = \frac{e(\text{Def}(C)^{\text{mov}})}{e(\text{Ob}(u)^{\text{mov}})}$$

Now $\text{Def}(C)^{\text{mov}}$ comes from resolving the node v of $\Sigma_1 \cup \Sigma_2$. By the *boundary lemma* [17, 25.2.2], the relevant normal bundle associated to this smoothing is $T_v \Sigma_1 \otimes T_v \Sigma_2$. The action on these tangent spaces is induced by the action on the image under the isomorphisms $U_{10} : \Sigma_1 \rightarrow \mathbb{P}^1 \times 0, U_{01} : \Sigma_2 \rightarrow \infty \times \mathbb{P}^1$ for Γ_{10} , and similarly for U_{01} . Recall that if μ_0, μ_1 are weights for a torus action on \mathbb{P}^1 then the weights for $T_0 \mathbb{P}^1, T_\infty \mathbb{P}^1$ are respectively $\mu_0 - \mu_1, \mu_1 - \mu_0$. So the weight for the action on the above tensor for U_{10}, U_{01} respectively are:

$$0 + (\alpha_0 - \alpha_1) \quad (\alpha_1 - \alpha_0) + 0.$$

Finally, consider $\text{Ob}(u)^{\text{mov}}$. The only contributions come from $\mathcal{O}(-1, -1)$. The normalizing sequence for the node for $u = U_{10}$ is:

$$0 \rightarrow u^* \mathcal{O}(-1, -1) \rightarrow \mathcal{O}_{\Sigma_1}(-1) \oplus \mathcal{O}_{\Sigma_2}(-1) \rightarrow u^* \mathcal{O}_{p_{10}}(-1, -1) \rightarrow 0$$

Taking the LES in cohomology, using that $H^1(\mathbb{P}^1, \mathcal{O}(-1)) = 0$, we deduce:

$$\begin{aligned} \text{Ob}(u)^{\text{mov}} &= H^0(\Sigma, u^* \mathcal{O}_{p_{10}}(-1, -1)) \equiv H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{p_{10}}(-1, -1)) \\ &= \mathcal{O}(-1, -1)|_{p_{10} \in \mathbb{P}^1 \times \mathbb{P}^1}. \end{aligned}$$

In general, the action on $\mathcal{O}(-1, -1)$ induced by the \mathbb{T} -action on $\mathbb{P}^1 \times \mathbb{P}^1$ has weights $-\rho_{ij}$ if ρ_{ij} are the weights for $\mathbb{P}^1 \times \mathbb{P}^1$ indexed by its fixed points p_{ij} . In our case, we obtain weight $-\alpha_0$. Similarly, for U_{01} we obtain $\mathcal{O}(-1, -1)|_{p_{01}}$ and weight $-\alpha_1$.

$$A_0 = \frac{-\alpha_0}{\alpha_0 - \alpha_1} + \frac{-\alpha_1}{\alpha_1 - \alpha_0} = \frac{-\alpha_0 + \alpha_1}{\alpha_0 - \alpha_1} = -1.$$

A_0 actually needs to be rescaled by $-n = -1$, because the perturbed P intersects the zero section in $-n[\text{pt}]$. This will become clearer in the next proof. \square

Definition 62. Let $\tau_{a,n}$ denote the coefficient of x^a in the degree $n-1$ polynomial

$$\prod_{\substack{A \geq 1, B \geq 1 \\ A+B=n}} (Ax + B),$$

and define $\tau_{0,1} = 1$. Observe that $\sum_a \tau_{a,n} = \prod (Ax + B)|_{x=1} = \prod n = n^{n-1}$.

In characteristic 2 and odd n , $\prod (Ax + B) \equiv x^{\frac{n-1}{2}}$, so $\tau_{a,n} \equiv 0$ except for $\tau_{\frac{n-1}{2}, n} = 1$, and $\sum_a \tau_{a,n} \equiv 1$. For even n , $\tau_{a,n} \equiv 0$ except when $n = 2$: $\tau_{0,2} = \tau_{1,2} = 1$.

Theorem 63. For $\mathcal{O}(-n) \rightarrow \mathbb{P}^m$, $A_a = n^2 \tau_{a,n}$ (assuming $n < 1 + m$).

Proof. $A_a = \text{GW}_{0,2,(1,1)}^{E_g}(j_{z_0} F_{a+1}, j_{z_\infty} \mathbb{P}^{n-a})$. We choose $F_{a+1} = \pi_M^{-1}(\mathbb{P}^a)$ where $\mathbb{P}^a \subset \mathbb{P}^m$ involves only the first $a+1$ homogeneous coordinates. We perturb \mathbb{P}^{n-a} vertically so that it intersects the zero section in $-n[\mathbb{P}^{n-a-1}]$. We can ensure $\mathbb{P}^{n-a-1} \subset \mathbb{P}^m$ involves only the last $n-a$ homogeneous coordinates (notice $\mathbb{P}^a, \mathbb{P}^{n-a-1}$ do not intersect since $n < 1 + m$). We will calculate the contribution of each $+\mathbb{P}^{n-a-1}$ separately, so we need to rescale the final answer by $-n$.

The $\mathbb{T} = (\mathbb{C}^*)^{m+1}$ action on $\mathbb{P}^1 \times \mathbb{P}^m$ is analogous to (4) above, acting on \mathbb{P}^m with weights $\alpha_0, \dots, \alpha_m$. The fixed points in \mathbb{P}^m are labeled q_ℓ having entry $w_\ell = 1$ and all other entries $w_r = 0$. Abbreviate $p_0 = [1 : 0] = 0, p_1 = [0 : 1] = \infty \in \mathbb{P}^1$ and

$$p_{k\ell} = p_k \times q_\ell \in \mathbb{P}^1 \times \mathbb{P}^m$$

where $k = 0, 1$ and $\ell = 0, 1, \dots, m$. Among this ℓ indexing, we reserve the letter $i = 0, 1, \dots, a$ and the letter $j = m - (n - a - 1), \dots, m$. These labels index the fixed points $q_i \in \mathbb{P}^a \subset \mathbb{P}^m$ and $q_j \in \mathbb{P}^{n-a-1} \subset \mathbb{P}^m$.

The open part of the moduli space \mathcal{M} are holomorphic $u : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^m \subset E_g$ satisfying the intersection conditions

$$u(0) \in p_0 \times \mathbb{P}^a \quad u(\infty) \in p_1 \times \mathbb{P}^{n-a-1}.$$

So they are lines which are geometrically determined by the intersection conditions. The union of all points lying on such lines spans a certain $\mathbb{P}^n \subset \mathbb{P}^m$.

Explicitly, given $[\vec{x}] \in \mathbb{P}^a$, $[\vec{y}] \in \mathbb{P}^{n-a-1}$, the line $[z_0 : z_1] \mapsto [z_0 \vec{x} + z_1 \vec{y}]$ is the unique geometric line through $[\vec{x}], [\vec{y}]$. However, the parametrization is not canonical: there is a \mathbb{P}^1 -freedom to reparametrize. Thus \mathcal{M} is a \mathbb{C}^* -bundle over $\mathbb{P}^a \times \mathbb{P}^{n-a-1}$. The compactification $\overline{\mathcal{M}}$ to a \mathbb{P}^1 -bundle is just fiberwise the same as

the one we did for the $m = n = 1$ case: a bubble appears in the M -fiber of E_g over p_0 or over p_1 . The universal curve is again a blow-up:

$$\begin{array}{c} \mathcal{C} = \text{Bl}(\mathbb{P}^1 \times \mathbb{P}^n, \mathbb{P}^a \sqcup \mathbb{P}^{n-a-1}) \xrightarrow{f=\text{ev}_3} \mathbb{P}^1 \times \mathbb{P}^n \subset \mathbb{P}^1 \times \mathbb{P}^m \subset E_g \\ \downarrow \pi \\ \overline{\mathcal{M}} = (\mathbb{P}^1\text{-bundle over } \mathbb{P}^a \times \mathbb{P}^{n-a-1}) \end{array}$$

The induced \mathbb{T} -action on $\overline{\mathcal{M}}$ is analogous to (4). The fixed stable maps $u : \Sigma_1 \cup \Sigma_2 \rightarrow \mathbb{P}^1 \times \mathbb{P}^m$ are indexed U_{1ij} and U_{ij0} , meaning: $u(0) = p_{0i}$, $u(\infty) = p_{1j}$,

$$U_{1ij}(\text{node}) = p_{1i}, \quad U_{ij0}(\text{node}) = p_{0j}.$$

The graphs $\Gamma_{1ij}, \Gamma_{ij0}$ have two edges and labelling $0i, 1i, 1j$ and $0i, 0j, 1j$ respectively. In this setup, $T^v E_g = T\mathbb{P}^m \oplus \mathcal{O}(-1, -n)$ and

$$\begin{aligned} (U_{1ij}^* T^v E_g)|_{\Sigma_1} &\cong T_{q_i} \mathbb{P}^m \oplus \mathcal{O}(-1), & (U_{1ij}^* T^v E_g)|_{\Sigma_2} &\cong \mathcal{O}(2) \oplus \mathcal{O}(1)^{m-1} \oplus \mathcal{O}(-n), \\ (U_{ij0}^* T^v E_g)|_{\Sigma_2} &\cong T_{q_j} \mathbb{P}^m \oplus \mathcal{O}(-1), & (U_{ij0}^* T^v E_g)|_{\Sigma_1} &\cong \mathcal{O}(2) \oplus \mathcal{O}(1)^{m-1} \oplus \mathcal{O}(-n) \end{aligned}$$

where $\mathcal{O}(2) \oplus \mathcal{O}(1)^{m-1}$ comes from pulling back $T\mathbb{P}^m$.

$\text{Def}(C)^{\text{mov}}$ comes from resolving the node, giving opposite weights

$$(1) \quad \alpha_i - \alpha_j \quad \alpha_j - \alpha_i$$

respectively for U_{1ij}, U_{ij0} . This time, $\text{Def}(u)$ has moving parts because we can deform the image of the fixed marked points x_1, x_2 within $\mathbb{P}^a, \mathbb{P}^{n-a-1}$ respectively. This yields two summands: $T_{q_i} \mathbb{P}^a$ and $T_{q_j} \mathbb{P}^{n-a-1}$, which have weights

$$(2) \quad \alpha_i - \alpha_I \quad \alpha_j - \alpha_J$$

where $0 \leq I \leq a$, $I \neq i$ and $m - (n - a - 1) \leq J \leq m$, $J \neq j$.

For $\text{Ob}(u)^{\text{mov}}$ only $\mathcal{O}(-1, -n)$ contributes, the normalizing sequence for U_{1ij} is:

$$0 \rightarrow u^* \mathcal{O}(-1, -n) \rightarrow \mathcal{O}_{\Sigma_1}(-1) \oplus \mathcal{O}_{\Sigma_2}(-n) \rightarrow u^* \mathcal{O}_{p_{1i}}(-1, -n) \rightarrow 0$$

and taking the LES in cohomology we deduce

$$\begin{aligned} \text{Ob}(U_{1ij})^{\text{mov}} &= \mathcal{O}(-1, -n)|_{p_{1i} \in \mathbb{P}^1 \times \mathbb{P}^m} \oplus H^1(\Sigma_2, \mathcal{O}_{\Sigma_2}(-n)) \\ \text{Ob}(U_{ij0})^{\text{mov}} &= \mathcal{O}(-1, -n)|_{p_{0j} \in \mathbb{P}^1 \times \mathbb{P}^m} \oplus H^1(\Sigma_1, \mathcal{O}_{\Sigma_1}(-n)) \end{aligned}$$

The first summands yield the following weights for U_{1ij}, U_{ij0} respectively:

$$(3) \quad -n\alpha_i \quad -n\alpha_j.$$

We now seek the weights for the H^1 summands. We consider the case $u = U_{1ij}$. By Serre duality, $H^1(\Sigma_2, \mathcal{O}_{\Sigma_2}(-n)) \cong H^0(\Sigma_2, K_{\Sigma_2} \otimes \mathcal{O}_{\Sigma_2}(n))^{\vee}$. The weights for the canonical bundle $K_{\Sigma_2} = T^* \Sigma_2$ at p_{1i}, p_{1j} are respectively $\alpha_j - \alpha_i$ and $\alpha_i - \alpha_j$. The $\mathcal{O}_{\Sigma_2}(n)$ comes from pulling back $\mathcal{O}(1, n)$ via an embedding $\Sigma_2 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^m$, and the weights for $\mathcal{O}(1, n)$ at p_{1i}, p_{1j} are $n\alpha_i$ and $n\alpha_j$. The total weights on $K_{\Sigma_2} \otimes \mathcal{O}_{\Sigma_2}(n)$ are therefore $\alpha_j + (n-1)\alpha_i$ and $\alpha_i + (n-1)\alpha_j$.

Since $\deg(K_{\Sigma_2} \otimes \mathcal{O}_{\Sigma_2}(n)) = n - 2$, it follows [17, 27.2.3] that the weights on $H^0(\Sigma_2, K_{\Sigma_2} \otimes \mathcal{O}_{\Sigma_2}(n))^{\vee}$ are

$$(4) \quad \frac{a}{n-2}[\alpha_j + (n-1)\alpha_i] + \frac{b}{n-2}[\alpha_i + (n-1)\alpha_j] = A\alpha_i + B\alpha_j$$

where $a + b = n - 2$ and $a, b \geq 0$, and where we simplified the expression using $A = a + 1, B = b + 1$, so $A, B \geq 1$ and $A + B = n$.

Similarly, for U_{ij0} we get weights $A\alpha_i + B\alpha_j$.

We now apply virtual localization, so we calculate $\sum \frac{1}{e^{\mathbb{T}}(N_{\Gamma}^{\text{vir}})}$:

$$\sum_{i,j} \frac{[(-n\alpha_i) - (-n\alpha_j)] \prod_{A,B} (A\alpha_i + B\alpha_j)}{(\alpha_i - \alpha_j) \prod_I (\alpha_i - \alpha_I) \prod_J (\alpha_j - \alpha_J)} = -n \sum_{i,j} \frac{\prod (A\alpha_i + B\alpha_j)}{\prod (\alpha_i - \alpha_I) \prod (\alpha_j - \alpha_J)}$$

This is supposed to be an integer: this can be verified taking common denominators:

$$-n \frac{\sum_{i,j} \prod (A\alpha_i + B\alpha_j) \prod (\alpha_{\hat{i}} - \alpha_I) \prod (\alpha_{\hat{j}} - \alpha_J)}{\prod (\alpha_c - \alpha_d) \prod (\alpha_p - \alpha_q)}$$

where $c \neq d$ vary in $\{0, 1, \dots, a\}$; $p \neq q$ in $\{m - (n - a - 1), \dots, m\}$; \hat{i} in $\{0, 1, \dots, a\} \setminus i$; and \hat{j} in $\{m - (n - a - 1), \dots, m\} \setminus j$; and $A, B, I \neq \hat{i}, J \neq \hat{j}$ are as usual. One then needs to show that each factor on the denominator, such as $(\alpha_c - \alpha_d)(\alpha_d - \alpha_c)$, divides the numerator. This amounts to checking that the numerator vanishes to order 2 when putting $\alpha_c = \alpha_d$.

To find that integer value we consider the fraction as a Laurent polynomial in one variable, say α_0 , with coefficients in the ring $\mathbb{Z}(\alpha_1, \dots, \alpha_m)$. Since only the α_0^0 term survives, we can drop all terms of different order. The denominator $\prod (\alpha_i - \alpha_I)$ of the original sum has order α_0^a , and the numerator has no α_0 terms unless $i = 0$. So we can put $i = 0$. Now, we can let $\alpha_0 \in \mathbb{R}$ and $\alpha_0 \rightarrow \infty$, so only this survives:

$$-n \sum_j \frac{\alpha_j^{n-a-1} \tau_{a,n}}{\prod (\alpha_j - \alpha_J)}$$

Let $\alpha_m \in \mathbb{R}$ and $\alpha_m \rightarrow \infty$, so only the $j = m$ term survives: $-n\tau_{a,n}$. Finally, recall from the beginning of the proof that we need to rescale the final answer by $-n$. \square

Proof of Theorem 5. Over characteristic 0 (Remark 65), in Lemma 59: $a_N = (-1)^N n^{N-1} (\sum t_{a,n})t = (-1)^N n^{1+m}t$ by the previous Theorem. For $n < 1 + \frac{m}{2}$,

$$\begin{aligned} QH^*(M) &= \Lambda[c_Q]/(c_Q^{1+m} + (-1)^N n^{1+m} t c_Q^n) = \Lambda[\omega_Q]/(\omega_Q^{1+m} + n^n t \omega_Q^n) \\ SH^*(M) &= \Lambda[c_Q]/(c_Q^N + (-1)^N n^{1+m} t) = \Lambda[\omega_Q]/(\omega_Q^N + n^n t). \end{aligned}$$

For $n < 1 + m$, $QH^*(M) = \Lambda[\omega_Q]/(\omega_Q^{1+m} + n^n t \omega_Q^n + \dots)$ may have lower order correction terms from $d \geq 2$ contributions, but we still deduce $SH^*(M) \neq 0$. For n even, vanishing in characteristic 2 is because $c_1(L)[\mathbb{P}^1] = -n \equiv 0 \pmod{2}$. \square

10. GENERAL THEORY FOR NEGATIVE LINE BUNDLES $M = \text{Tot}(\pi_M : L \rightarrow B)$

Theorem 64. For $M = \text{Tot}(\pi_M : L \rightarrow B)$ (satisfying weak⁺ monotonicity), the constant sections are regular for $\hat{J} = \begin{bmatrix} j & 0 \\ 0 & i \end{bmatrix}$ and they determine

$$r_{\hat{g}}(1) = (1 + \lambda_+) \pi_M^* c_1(L)$$

where λ_+ lies in the subring $\Lambda_+^0 \subset \Lambda$ generated by the $\pi_2(M)$ -classes with $\omega > 0$ and $c_1(TM) = 0$ (for monotone M , $\lambda^+ = 0$). In particular, $(1 + \lambda^+)$ is a unit of Λ , so for the purposes of calculating $SH^*(M)$, we may rescale $r_{\hat{g}}(1) = \pi_M^* c_1(L)$.

Proof. Consider the dimension of the moduli space of sections

$$\dim_{\mathbb{C}} \mathcal{S}(j, \hat{J}, \gamma + S_{\hat{g}}) = b + c$$

where $b = \dim_{\mathbb{C}} B$ and $c = c_1(TM, \omega)(\gamma)$. Since M is weak, $c \geq 0$ or $c \leq -b$. Since $r_{\hat{g}}(1)$ sweeps an lf cycle, we may assume $1 \leq b + c \leq b$ ([pt] is a boundary if

cycle, and we cannot sweep $[M]$ by the maximum principle). Combining: $c = 0$. In the monotone case, this implies the sections are constant, so $\gamma = 0$. In general, constant sections of E_g are regular for the integrable \hat{J} by mimicking Lemma 46:

$$\text{coker } \bar{\partial} = H^1(\mathbb{P}^1, \mathcal{O}(\mathbb{C}^{\oplus \dim_{\mathbb{C}} B}) \oplus \mathcal{O}(-1)) \cong H^0(\mathbb{P}^1, \mathcal{O}(-2)^{\oplus \dim_{\mathbb{C}} B} \oplus \mathcal{O}(-1))^{\vee} = 0.$$

Constants sweep the lf cycle $\text{ev}_{\infty}(\mathcal{S}(j, \hat{J}, S_{\hat{g}})) = [B]$. Consider a 1-cycle $\alpha \subset B$:

$$\alpha \bullet_M [B] = \alpha \bullet_B (\text{zero set of a generic } C^{\infty}\text{-section}).$$

The zero set is obtained by perturbing $[B]$, it represents $\text{PD}_B(c_{\text{top}}(L))$ in B . Pull-back $\pi_M^*: H^*(B) \rightarrow H^*(M)$ in cohomology is Poincaré dual to taking pre-images $\pi_M^{-1}: H_*(B) \rightarrow H_{*+2}^{lf}(M)$ (Bott-Tu [3, Sec.6]). So $\pi_M^{-1}\text{PD}_B(c_1(L)) = \text{PD}_M(\pi_M^*c_1(L))$.

In the non-monotone case, it may happen that $c = 0$ but $\omega(\gamma) > 0$. The only lf $2m$ -cycles supported near the zero section are multiples of $[B]$ (generator of $H_{2m}(B)$), so $\mathcal{S}(j, \hat{J}, \gamma + S_{\hat{g}})$ is a multiple of $[B]$. These determine λ_+ . \square

Example. For $\mathcal{O}(-n) \rightarrow \mathbb{P}^m$: the lf cycle $[\mathbb{P}^m]$ when perturbed vertically will intersect the zero section in $-n[\mathbb{P}^{m-1}]$, so the intersection number $\mathbb{P}^1 \bullet [\mathbb{P}^m] = -n$ in M , so $[\mathbb{P}^m] = -n[F_m] \in H_{2m}^{lf}(M)$, so $\text{PD}[\mathbb{P}^m] = -n[\omega_{\mathbb{P}^1}] = c_1(\mathcal{O}(-n)) \in H^2(M)$.

Remark. $\lambda_+ \neq 0$ can occur only for non-monotone M , and a base B admitting a holomorphic map $v: \mathbb{P}^1 \rightarrow B$ through any given point with $c_1(TB)(v) = n\omega_B(\pi_M v)$.

Remark 65 (Orientation Signs and $\text{char}(\Lambda) \neq 2$). For regular integrable complex structures the moduli spaces of holomorphic curves are canonically oriented (see [15, Rmk 3.2.6, p.51]) and the 0-dimensional ones always contribute with sign +1. So the dominant term $\pi_M^*c_1(L)$ in $r_{\hat{g}}(1)$ is correct also if we work over Λ of characteristic zero (e.g. in 2.6 replace $\mathbb{Z}/2$ by \mathbb{Z} or \mathbb{Q}). Lemma 13, $QH^*(M) \cong HF^*(H_0)$, still holds (orientation signs for $SH^*(M)$ and its product structure were constructed by the author in [21]). Therefore Theorem 1 holds also over characteristic zero.

11. NEGATIVE VECTOR BUNDLES

Definition 66. A complex vector bundle $E \rightarrow B$ over a closed symplectic manifold (B, ω_B) is negative if E admits a Hermitian metric, and some Hermitian connection whose curvature $\mathcal{F} \in \Omega^2(B, \mathfrak{u}(E))$ satisfies

$$\frac{i}{2\pi} \mathcal{F}(v, J_B v) < 0$$

for all $v \neq 0 \in TB$ (meaning that is a negative definite Hermitian endomorphism of E), for all almost complex structures J_B compatible with ω_B .

Lemma 67. The total space M of a negative vector bundle $E \rightarrow B$ is symplectic (but non-conical) for the form $\omega = \pi_E^* \omega_B + \Omega$, defined using the connection above:

$$\begin{aligned} \Omega &= \frac{1}{\pi} (\text{area form}) \text{ on vertical vectors (in a local unitary frame for } E) \\ \Omega_{(b,w)}(\cdot, \cdot) &= \frac{1}{2\pi i} w^\dagger \mathcal{F}_{(d\pi_E, d\pi_E)} w \text{ on horizontal vectors, for } w \neq 0 \\ \Omega_{(b,0)}(TB, \cdot) &= 0 \\ \Omega_{(b,w)}(h, v) &= 0 \text{ if } h \text{ is horizontal, and } v \text{ is vertical.} \end{aligned}$$

and ω is compatible with $J = J_B \oplus i$ acting on $T^{\text{horiz}} E \oplus T^{\text{vert}} E$.

Proof. We start by a standard trick from algebraic geometry. Consider the (complex) projectivization $\mathbb{P}(E)$ of E . Let $L = \mathcal{O}(-1) \rightarrow \mathbb{P}(E)$ be the tautological line bundle, so L is just $\mathcal{O}(-1) \rightarrow \mathbb{P}^{\text{rank}(E)-1}$ over each fibre of $\mathbb{P}(E) \rightarrow B$:

$$\begin{array}{ccc} L & E & \mathcal{O}(-1) & E_b \\ \pi_L \downarrow & \downarrow \pi_E & \downarrow & \downarrow \\ \mathbb{P}(E) & \xrightarrow{\pi_E} B & \mathbb{P}^{\text{rank } E - 1} & \longrightarrow b \end{array}$$

By Leray-Hirsch (see [11, Appendix A] or [1, p.134]): $H^*(\mathbb{P}(E))$ is a free $H^*(B)$ -module via $\pi_{\mathbb{P}}^*$, generated by $1, c_1(L), \dots, c_1(L)^{\text{rank}(E)-1}$.

A horizontal distribution for E yields a horizontal distribution for L spanned by the horizontal vectors of E and the horizontal vectors of each $\mathcal{O}(-1) \rightarrow \mathbb{P}^{\text{rank}(E)-1}$.

Suppose E is negative, and pick a Hermitian metric and connection as in the definition. It is carefully proved in Oancea [16, Sec.3.4] that this determines a canonical Hermitian metric and Hermitian connection on $\mathbb{P}(E)$ and L , and that this determines a canonical symplectic form on $\text{Tot}(\mathbb{P}(E))$ given by the curvature

$$\omega_{\mathbb{P}} = -\frac{i}{2\pi} \mathcal{F}^L \quad (\text{hence } c_1(L) = -[\omega_{\mathbb{P}}])$$

which restricts to the normalized Fubini-Study form on each $\mathbb{P}^{\text{rank}(E)-1}$. So L is a negative line bundle. Explicitly, the curvature at $(b, [w])$, where $[w] = \mathbb{C}w$ is a line in the fibre E_b , is

$$\begin{aligned} \mathcal{F}_{(\cdot, \cdot)}^L &= \frac{1}{r^2} w^\dagger \mathcal{F}_{(d\pi_{\mathbb{P}}, d\pi_{\mathbb{P}})}^E w && \text{on horizontal vectors of } \mathbb{P}(E) \rightarrow B \\ &&& (\text{identifiable with horizontal vectors of } E) \\ \mathcal{F}_{(\cdot, \cdot)}^L &= \mathcal{F}_{(\cdot, \cdot)}^{\mathcal{O}(-1) \rightarrow \mathbb{P}^{\text{rank}(E)-1}} && \text{on } \ker d\pi_{\mathbb{P}}, \text{ the vertical vectors of } \mathbb{P}(E) \rightarrow B \\ &&& (\text{identifiable with } (\mathbb{C}w)^\perp \subset T_{(b,w)}^{\text{vert}} E \equiv E_b) \end{aligned}$$

and mixed terms vanish (the horizontal vectors of $\mathbb{P}(E) \rightarrow B$ are $\omega_{\mathbb{P}}$ -orthogonal to the vertical ones). Let $\theta^L, \Omega^L = d(r^2 \theta^L)$ be as in 7.1, so $d\theta^L = \frac{1}{2\pi i} \pi_L^* \mathcal{F}^L$. Let

$$\tau : \text{Tot}(E \setminus 0_E) \rightarrow \text{Tot}(L \setminus 0_L)$$

be the tautological isomorphism (a point on the right is a choice of complex line in a fibre of E together with a choice of vector in that line, so it is point in $\text{Tot}(E)$).

Outside the zero section of E , define

$$\theta = \tau^*(\theta^L), \quad \Omega = \tau^*(\Omega^L) = d(r^2 \theta)$$

The angular form $\theta_w = \frac{1}{2\pi r^2} \langle iw, \cdot \rangle$ (taking the vertical component of \cdot and using the Hermitian metric) is $U(\text{rank } E)$ -invariant, and fiberwise $\Omega = (\text{area form})/\pi$. As in 7.1, Ω extends over the zero section. To finish proving Ω is as claimed, we use Lemma 39: for $w \neq 0$, and horizontal vectors h, h' of E ,

$$d\theta_w \cdot (h, h') = \theta_{[w], w}^L \left(\frac{1}{r^2} w^\dagger \mathcal{F}_{(d\pi_E h, d\pi_E h')}^E w \right) = \frac{1}{2\pi i r^2} w^\dagger \mathcal{F}_{(d\pi_E h, d\pi_E h')}^E w$$

using that \mathcal{F}^E is skew-Hermitian and $d\pi_{\mathbb{P}} d\pi_L d\tau = d\pi_E$.

We now prove $\omega = \pi_E^* \omega_B + \Omega$ is symplectic. Let J_B be ω_B -compatible, then we obtain an almost complex structure $J = J_B \oplus i$ on $T^{\text{horiz}} E \oplus T^{\text{vert}} E = TE$ (J_B canonically lifts to an action on horizontal vectors). On $h \neq 0 \in T_{(b,w)}^{\text{horiz}} E$,

$$\omega(h, Jh) = \omega_B(d\pi_E h, J_B d\pi_E h) + \frac{1}{2\pi i} w^\dagger \mathcal{F}_{(d\pi_E h, J_B d\pi_E h)} w > 0$$

using negativity of E (omitting the second term if $w = 0$). On $\ker d\pi_E$, Ω is the area form so it is symplectic and i -compatible. So ω is symplectic and J -compatible. \square

Remark. For $\text{rank}_{\mathbb{C}} E \geq 2$ any conical ω would be exact by the LES for the pair: $0 = H^3(E, E \setminus 0) \rightarrow H^2(E) \rightarrow H^2(E \setminus 0)$. So the zero section would not be symplectic.

Remarks about the maximum principle.

Let ω, J be as in Lemma 67. Since ω is no longer conical, we need to justify the maximum principles which relied on exactness at infinity.

For holomorphic curves, locally $u : \mathbb{P}^1 \supset U \rightarrow B \times \mathbb{C}^{\text{rank } E} \rightarrow \mathbb{C}^{\text{rank } E}$ is holomorphic (using $J = J_B \oplus i$), so it lies in the zero section unless it is constant.

For Floer trajectories and continuations, we use Hamiltonians which are radial in r at infinity: $H = h(r)$ where $r(b, w) = |w|$ is defined using the Hermitian metric. In a local trivialization for E obtained using a local unitary frame, r becomes the standard norm on the fibre $E_b \cong \mathbb{C}^{\text{rank } E}$ and the Hamiltonian orbits lie in fibres and are the usual Hamiltonian orbits for $(\mathbb{C}^{\text{rank } E}, \frac{1}{\pi}(\text{area form}))$ (in particular $X_H, JX_H = iX_H$ are vertical vectors, and in the region where $H = h(r)$ the slope of h determines the existence/non-existence of orbits). The maximum principle relied on finding a maximum principle for r for local solutions, so we can assume $u : \mathbb{R} \times S^1 \supset U \rightarrow B \times \mathbb{C}^{\text{rank } E}$. But $r : M \rightarrow \mathbb{R}$ locally factors through the $\mathbb{C}^{\text{rank } E}$ so the maximum principles reduce to the known maximum principles for $(\mathbb{C}^{\text{rank } E}, \frac{1}{\pi}(\text{area form}))$. For sections of E_g one argues similarly.

For Floer continuation solutions u , the monotonicity lemma still holds (using that $\text{Tot}(E)$ has bounded geometry at infinity).

For transversality, one may need to perturb $J = J_B \oplus i$. A trick we explain in the Appendix of [21] shows how a Gromov compactness argument can be used to deduce that the maximum principle for a perturbed J' holds if it held for J , provided that J' is inductively chosen sufficiently close to J on each of a collection of exhausting compacts, say $(r \leq m) \subset M$ for $m = 1, 2, \dots$. To achieve transversality one makes arbitrarily small perturbations of J' locally, so this freedom in the choice of J' suffices. Thus we can guarantee both compactness and transversality results for J' .

Lemma 68. $I(\tilde{g}) = \text{rank}_{\mathbb{C}} E$ for $g_t = e^{2\pi i t}$ acting by rotation in the fibres of E , lifted canonically to the \tilde{g} which fixes constants on the zero section.

Proof. This is proved as in Lemma 45: using a local unitary frame for E_b , $\ell(t)$ is the identity on the $T_b B$ factor and rotation by $e^{2\pi i t}$ of the fibre factor $E_b \cong \mathbb{C}^{\text{rank } E}$. So $t \mapsto \det(I \oplus e^{2\pi i t} I) = e^{2\pi i t \cdot \text{rank}_{\mathbb{C}} E}$ is $\text{rank}_{\mathbb{C}} E$ in $H_1(S^1; \mathbb{Z}) \cong \mathbb{Z}$. \square

Theorem 69. For $M = \text{Tot}(\pi_M : E \rightarrow B)$ (satisfying weak⁺ monotonicity), the analogue of Theorem 64 holds using $\pi_M^* c_{\text{rank}_{\mathbb{C}} E}(E)$.

Proof. The dimension of the moduli space (using Definition 23)

$$\dim_{\mathbb{C}} \mathcal{S}(j, \hat{J}, \gamma + S_{\tilde{g}}) = \dim_{\mathbb{C}} M - \text{rank}_{\mathbb{C}} E + c_1(TM, \omega)(\gamma) = b + c.$$

M is weak, so $c \geq 0$ or $c \leq 1 - \dim_{\mathbb{C}} M = 1 - b - r$ (let $r = \text{rank}_{\mathbb{C}} E$). Since we need to sweep an lf cycle, we may assume $r \leq b + c \leq b + r - 1$ (since $H_*^{lf}(M) \cong H^{2b+2r-*}(B)$, and it cannot sweep $[M]$ by the maximum principle). So $c \geq 0$. But the only lf cycles of degree $\geq 2b$ supported near the zero section are multiples of $[B]$ ($H_k(B) = 0$ for $k > 2b$). So $c = 0$. The rest is as in the proof of Theorem 64.

Proving constants are regular: for the constant section $u(z) = (z, y)$, $(u^* T^v E_g)_z = T_y B \oplus \mathbb{C}^r$ with transition $(\text{id}, g_t^{\oplus r})$ over the equator of S^2 . Therefore

$$\text{coker } \bar{\partial} = H^1(\mathbb{P}^1, \mathcal{O}(\underline{\mathbb{C}}^{\oplus b}) \oplus \mathcal{O}(-1)^{\oplus r}) \cong H^0(\mathbb{P}^1, \mathcal{O}(-2)^{\oplus b} \oplus \mathcal{O}(-1)^{\oplus r})^{\vee} = 0. \quad \square$$

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